Formalized algebraic numbers: construction and first-order theory

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Formalization of mathematics

Mathematics is a game: mathematicians must follow "rules" to convince their colleagues.

E.g.

$$\frac{\text{Garfield is a cat} \quad \text{all cats have four legs}}{\text{Garfield has four legs}}$$

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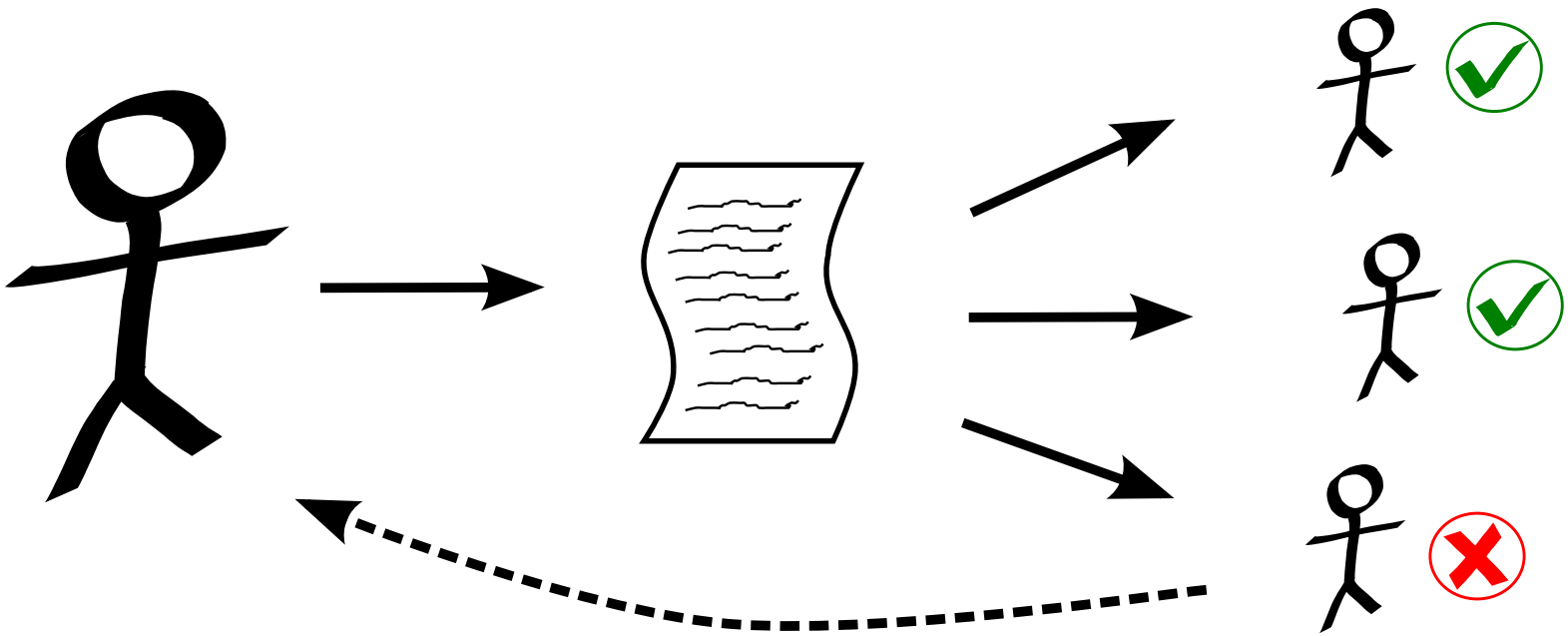
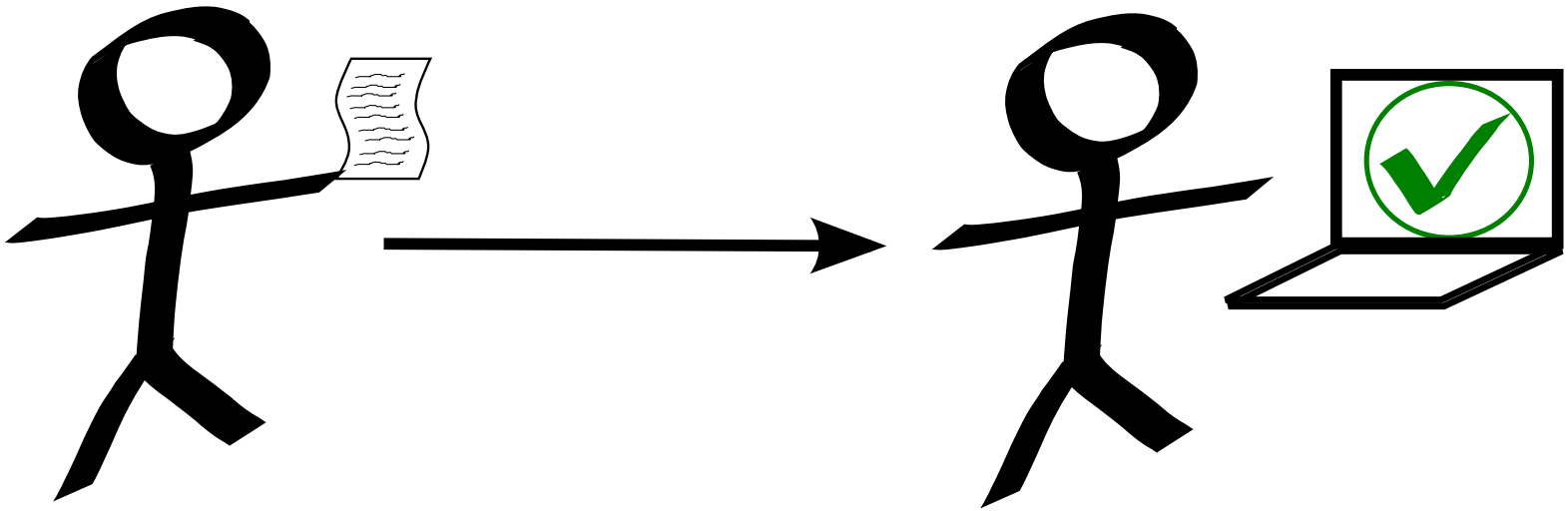
$$\frac{t : T \qquad \forall x : T, P(x)}{P(t)}$$

Paper proof



Computer checked proof



Feit-Thompson Theorem

Statement

Finite groups of odd order are solvable

- First proof: Feit and Thompson (1962)

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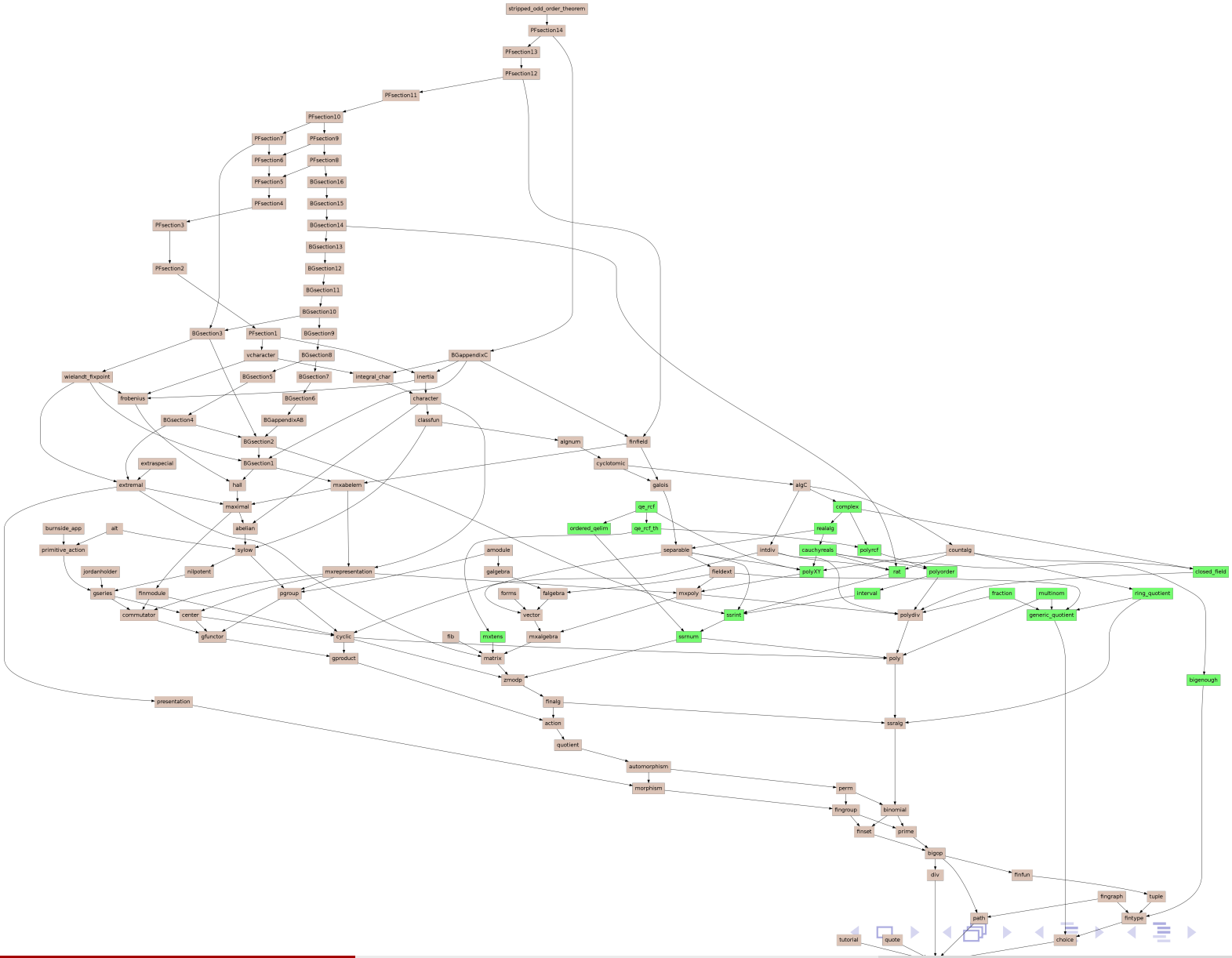
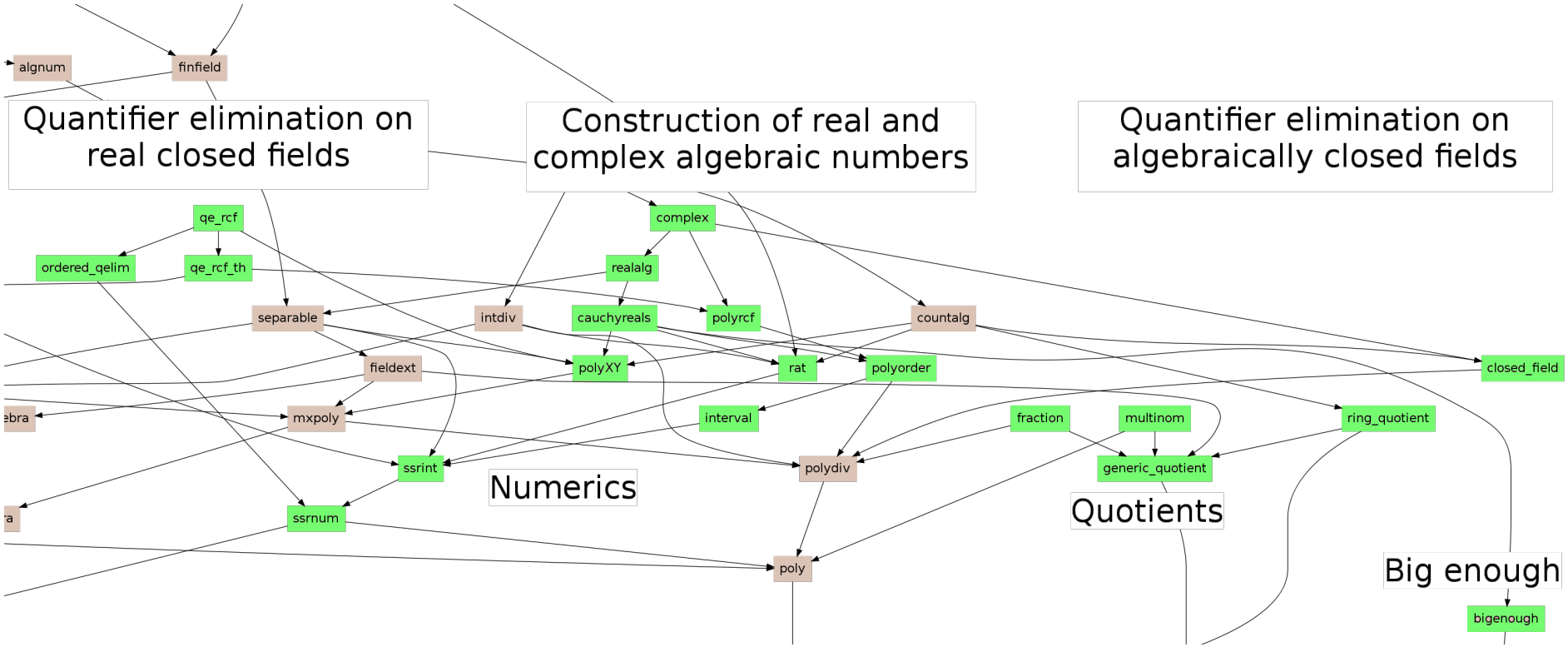
- Computer checked: Mathematical Components (September 2012)

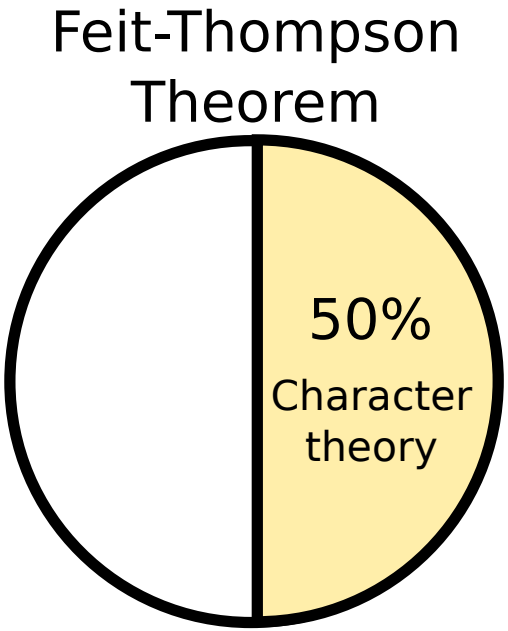
Mathematical Components project files



My contributions to the project



Complex numbers in Feit-Thompson Theorem



$$\chi_g : \mathbb{C}$$

$$\|\chi_g\| : \mathbb{R}$$

$$\|\chi_g\| > \frac{8}{15} > \frac{1}{2}$$

Outline

reals

complex := reals $[i]$

$\Leftrightarrow$ FTA

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- Factoring the theory of structures with order and norm
- Construction of real algebraic numbers
- The first-order theory of real and algebraic numbers is decidable (through quantifier elimination).

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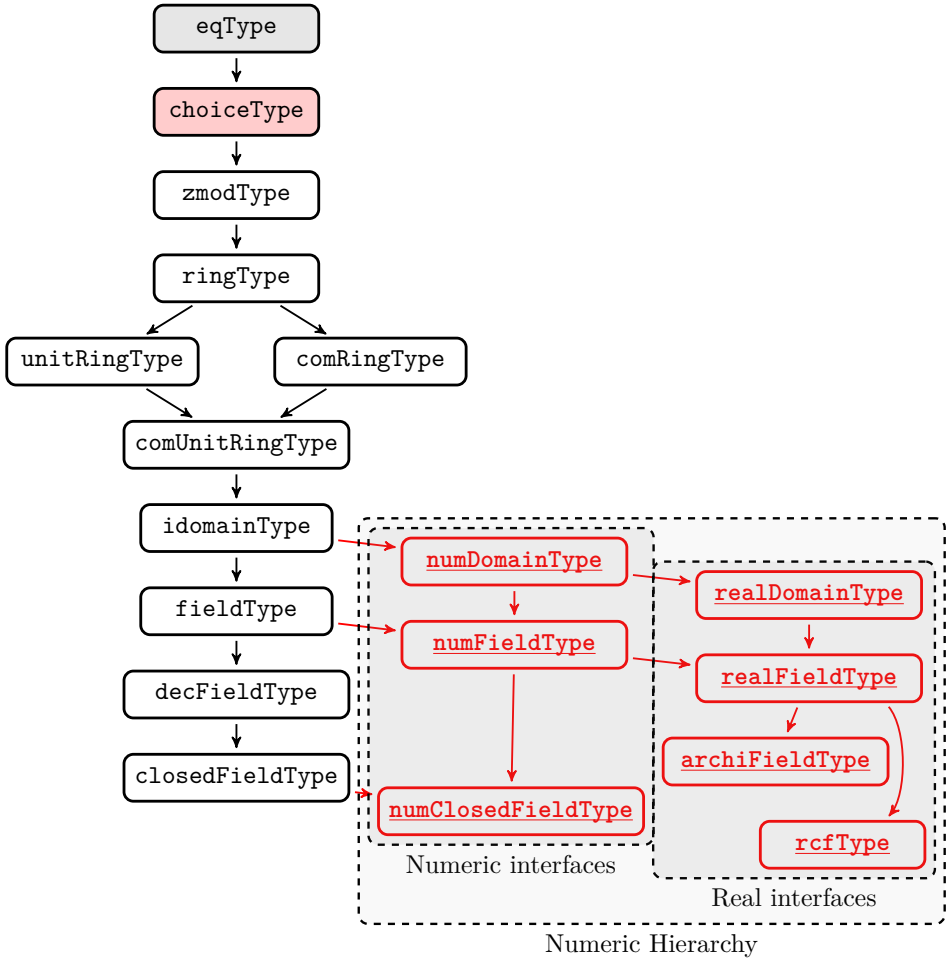
Interfaces


eqType
choiceType
zmodType
ringType
unitRingType
comRingType
comUnitRingType
idomainType
fieldType
decFieldType
closedFieldType
numDomainType
numFieldType
numClosedFieldType
realDomainType
realFieldType
archiFieldType
rcfType
Numeric interfaces
Real interfaces
Numeric Hierarchy

Why these interfaces ?

Goal:

- Factor and organize the theory of numbers from $\mathbb{Z}$ to algebraic numbers.
- Deal with the partial order complex algebraic numbers.

How?

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- Deal with the partial order complex algebraic numbers.

How?

- Reuse the packed class methodolgy (Garillot et al.)
- Based on the norm, not only $\leq$.
- Instances: integers, rationals, real and algebraic numbers.

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What are algebraic numbers?

(Complex) algebraic numbers are

- the **complex** *roots of polynomials* with coefficients in $\mathbb{Q}$.

Real *algebraic numbers* are:

the **real** *roots of polynomials* with coefficients in $\mathbb{Q}$.

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- $i, \sqrt{2} + i\sqrt{5}$ are algebraic
- π and e are not algebraic

Representations of real algebraic numbers

$x \in \mathbb{R}, P \in \mathbb{Q}[X]$

- ✓ operations (reconstruction of polynomial using resultant)
- ✗ countable type

$P \in \mathbb{Q}[X], [a, b]$

- ✗ operations
- ✓ countable type

Construction of real algebraic numbers

Goal:

- A countable type,
- Decidability of atoms ($=$ and $\leq$),
- RCF (intermediate value theorem for polynomials).

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- A formalization of Cauchy reals
- A quotient of type

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References: C.C., ITP 2012

Cauchy reals

Definition

$$(x_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}} \quad \text{and} \quad \mu_x : \mathbb{Q} \to \mathbb{N}$$

$$\text{such that} \quad \forall \varepsilon > 0, \forall i,\ j \geq \mu_x(\varepsilon), |x_i - x_j| \leq \varepsilon$$

- Formalized just what was needed
- Some $\varepsilon - \delta$ reasoning to formalize

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Big enough

Example

$$\text{if } x_n \to a \text{ and } y_n \to b, \text{ then } x_n y_n \to ab$$

Suppose $x_n \to a$ and $y_n \to b$.
Let ε be a positive rational. Show

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$$|x_n||y_n - b| + |x_n - a||b| \leq \varepsilon$$

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Suppose $x_n \to a$ and $y_n \to b$.
Let ε be a positive rational. Show

$$(1 + |a|)|y_n - b| + |x_n - a|(1 + |b|) \leq \varepsilon$$

because

$$|x_n - a| \leq 1$$

Big enough

Example

$$\text{if } x_n \to a \text{ and } y_n \to b, \text{ then } x_n y_n \to ab$$

Suppose $x_n \to a$ and $y_n \to b$.
Let ε be a positive rational. Show

$$|y_n - b| \leq \frac{\varepsilon}{2(1+|a|)} \quad \text{and} \quad |x_n - a| \leq \frac{\varepsilon}{2(1+|b|)}$$

Big enough

Goal:

- Do like in paper proofs.

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Usage:

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- Infer the *n a posteriori*.
- Based on Coq existential variables.

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Usage:

- More than 100 occurences in 3163 lines of code.

Quotient types

$$\mathrm{Type}/\equiv \longrightarrow \mathrm{Type}$$

Difficult problem in Constructive TT (Hoffman, Chicli at al., Courtieu).

$\Rightarrow$ We are interested in a particular case.

Particular case for quotienting

Conditions:

- Decidable equivalence.
- Countable type.

Consequence: possibility to select a unique element in each equivalence class.

Theory of quotient types

- Inference.
- Preservation of the ring structure while quotienting by an ideal.

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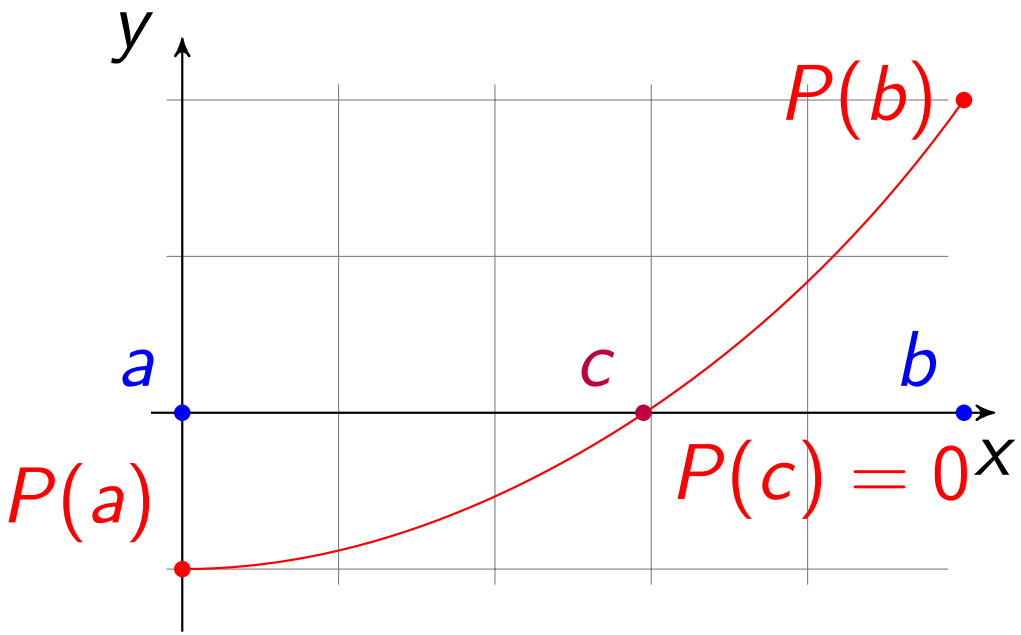
Definition of Real Closed Field

Field + order + intermediate value property for polynomials



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Real algebraic numbers **implement** the interface of real closed field

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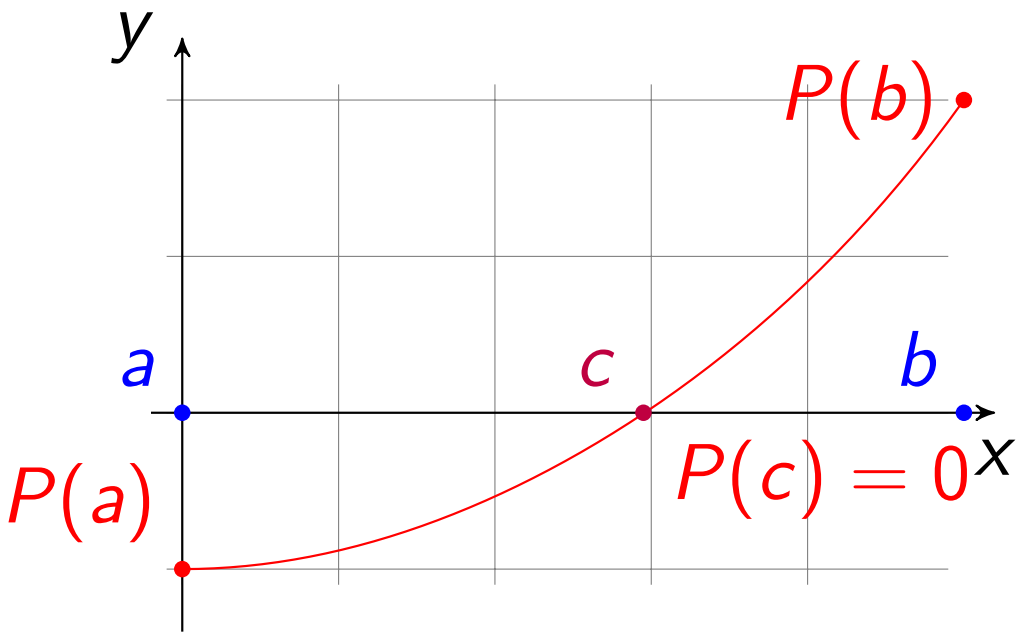

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The theory of Real Closed Fields

- Rolle, MVT, ...
- Infrastructure for intervals (membership, inclusion, splitting, ...)
- Neighborhoods

Classical reasoning on Real Closed Fields

- Decidability of the atoms ($=$ and $\leq$)

$\Rightarrow$ Decidability of simple formulas

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$\Rightarrow$ Classical reasoning

- Unless we can decide the validity of formulas

Quantifier elimination on real closed fields

Tarski (1948)

The first-order theory of real closed fields enjoys quantifier elimination.

Consequences:

- We can decide whether first-order formulas are valid.
- We can perform case analysis on quantifier formulas.

An example

```
R : rcfType
a : R
b : R
c : R
...
=======================
exists x, a * x ^ 2 + b * x + c = 0
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c : R
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```

- Prove the conclusion
- Provide the witness for `x`

An example

```
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a : R
b : R
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=======================
exists x, a * x ^ 2 + b * x + c = 0
```

- Prove the conclusion
- Eliminate the quantifier

An example (continued)

```
R : rcfType
a : R
b : R
c : R
...
======================
0 <= b ^ 2 - 4 * a * c
```

On Quantifier Elimination in Coq

Goal:

- Case reasoning on first-order formulas for ACF and RCF.

How?

- Deep embedding of first-order logic for RCF and ACF.
- Implement QE procedures and their formal proof.

References: CC and Mahboubi (Calculemus 2010, LMCS 2012).

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- Interface design (Numerics)
- Tools to mechanize tasks (Big enough, intervals, quotients)

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Good integration of the tools and the formalizations in the proof of Feit-Thompson Theorem.

Perspectives

- Generalize big enough numbers.
- Providing efficient implementations. Efficient algorithm are proved using naive ones.

$\Rightarrow$ application to fast real algebraic numbers

- An algebraic hierarchy based on types which admit uniqueness of identity proofs.
- Certifying the Cylindrical Algebraic Decomposition.

The end

Thank you for your attention.