# Tracking $p$-adic precision

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- Polynomial factorization with Hensel lemma ;
- Kedlaya's counting-point algorithm on hyperelliptic curves with $p$-adic cohomology ;

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# Norms over a field

## Definition

A norm over a field $K$ is a mapping $|\cdot| : K \to \mathbb{R}_+, x \mapsto |x|$ such that :

$$(i)\ |x| = 0 \Leftrightarrow x = 0;$$
$$(ii)\ |xy| = |x||y|;$$
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It is called **ultrametric** if :

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Let $K$ be a normed field. A norm over a $K$-vector space $E$ is a mapping $\|\cdot\| : E \to \mathbb{R}_+, x \mapsto \|x\|$ such that :

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We write :

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p,\ v_p(x) \geq 0\} = \{x \in \mathbb{Q}_p,\ |x|_p \leq 1\} = B'_{\mathbb{Q}_p}(0,1).$$

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$\mathbb{Z}_p$ is a sub-ring of $\mathbb{Q}_p$.

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And,

$$\ldots 4444,6_7 = \frac{4}{21}.$$

# Topology and ultrametricity

**Proposition**

*If $E$ is an ultrametric vector space, then **any** point in a ball of $E$ is its **center**.*

# Algebraic point of view

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## Remark

$$\begin{tikzcd} \mathbb{Q}_p \arrow[dr, dash] & & \mathbb{R} \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z}_p \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Z}_p \\ \mathbb{Z} \arrow[r, two heads] \arrow[u, hook] & \mathbb{Z}/p^k\mathbb{Z} \end{tikzcd}$$

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# Definition of the precision

## Finite-precision $p$-adics

Elements of $\mathbb{Q}_p$ can be written $\sum_{i=-l}^{+\infty} a_i p^i$, with $a_i \in [\![0, p-1]\!]$, $l \in \mathbb{Z}$ and $p$ a prime number.
While working with a computer, we usually only can consider the beginning of this power serie expansion: we only consider elements of the following form $\sum_{i=l}^{d-1} a_i p^i + O(p^d)$, with $l \in \mathbb{Z}$.

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## Example

The order of $3 * 7^{-1} + 4 * 7^0 + 5 * 7^1 + 6 * 7^2 + O(7^3)$ is 3.

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3. Lift the factors into $\mathbb{Z}/p^k\mathbb{Z}[X]$ (with *Hensel's lemma*) ;
4. If $p^k$ is big enough (*Mignotte's bound*), we can obtain a factorization over $\mathbb{Q}$ (up to the recombination of some factors).

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# $p$-adic precion *vs* real precision

The quintessential idea of the step-by-step analysis is the following :

**Proposition ($p$-adic errors don't add)**

*Indeed,*

$$(a + O(p^{k})) + (b + O(p^{k})) = a + b + O(p^{k}).$$

*That is to say, if $a$ and $b$ are known up to precision $O(p^k)$, then so is $a + b$.*

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**Remark**

It is quite the opposite to when dealing with real numbers, because of **Round-off error** :

$$(1 + 5 * 10^{-2}) + (2 + 6 * 10^{-2}) = 3 + 1 * 10^{-1} + 1 * 10^{-2}.$$

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## Proposition (division)

$$\frac{xp^a + O(p^b)}{yp^c + O(p^d)} = x * y^{-1}p^{a-c} + O(p^{min(d+a-2c,b-c)})$$

*In particular,*

$$\frac{1}{p^c y + O(p^d)} = y^{-1}p^{-c} + O(p^{d-2c})$$

# Optimality

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- If we apply Gaussian elimination, the average precision on $L$ is $O(p^{n-\frac{2d}{p-1}})$.
- If we study Cramer-style formulae, the intrinsic precision determined for $L$ is $O(p^{n-2\log_p(d)})$.

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# The Main lemma of $p$-adic differential precision

**Lemma**

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# The Main lemma of $p$-adic differential precision

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*Let $x \in \mathbb{Q}_p^n$. We assume that $f'(x)$ is **surjective**.*
*Then for any ball $B = B(0, r)$ **small enough**,*

$$f(x + B) = f(x) + f'(x) \cdot B.$$

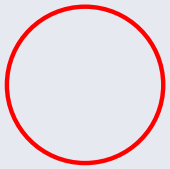
# Geometrical meaning

## Interpretation

$x+$

$+ \quad f(x)$

$B$



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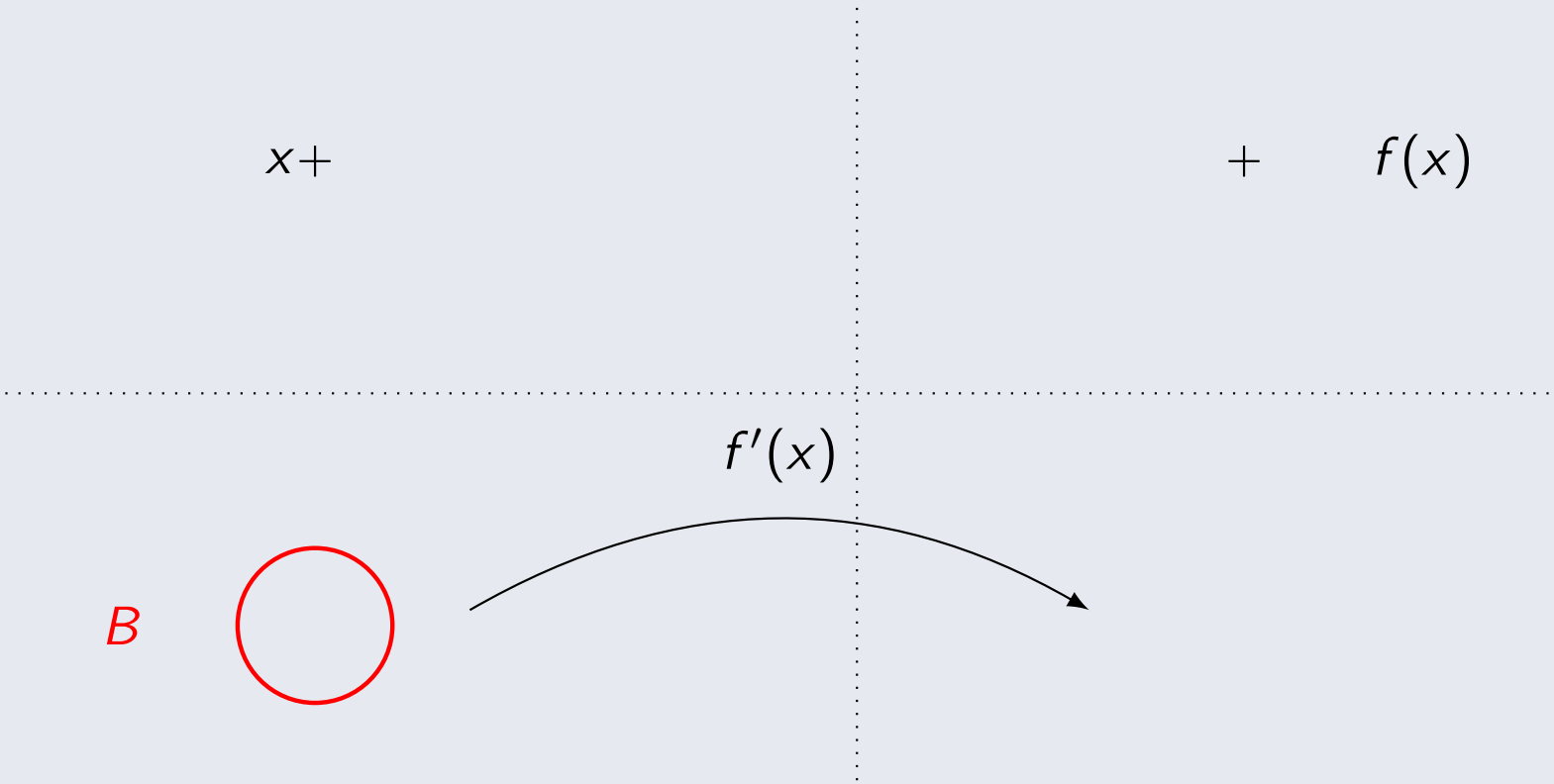

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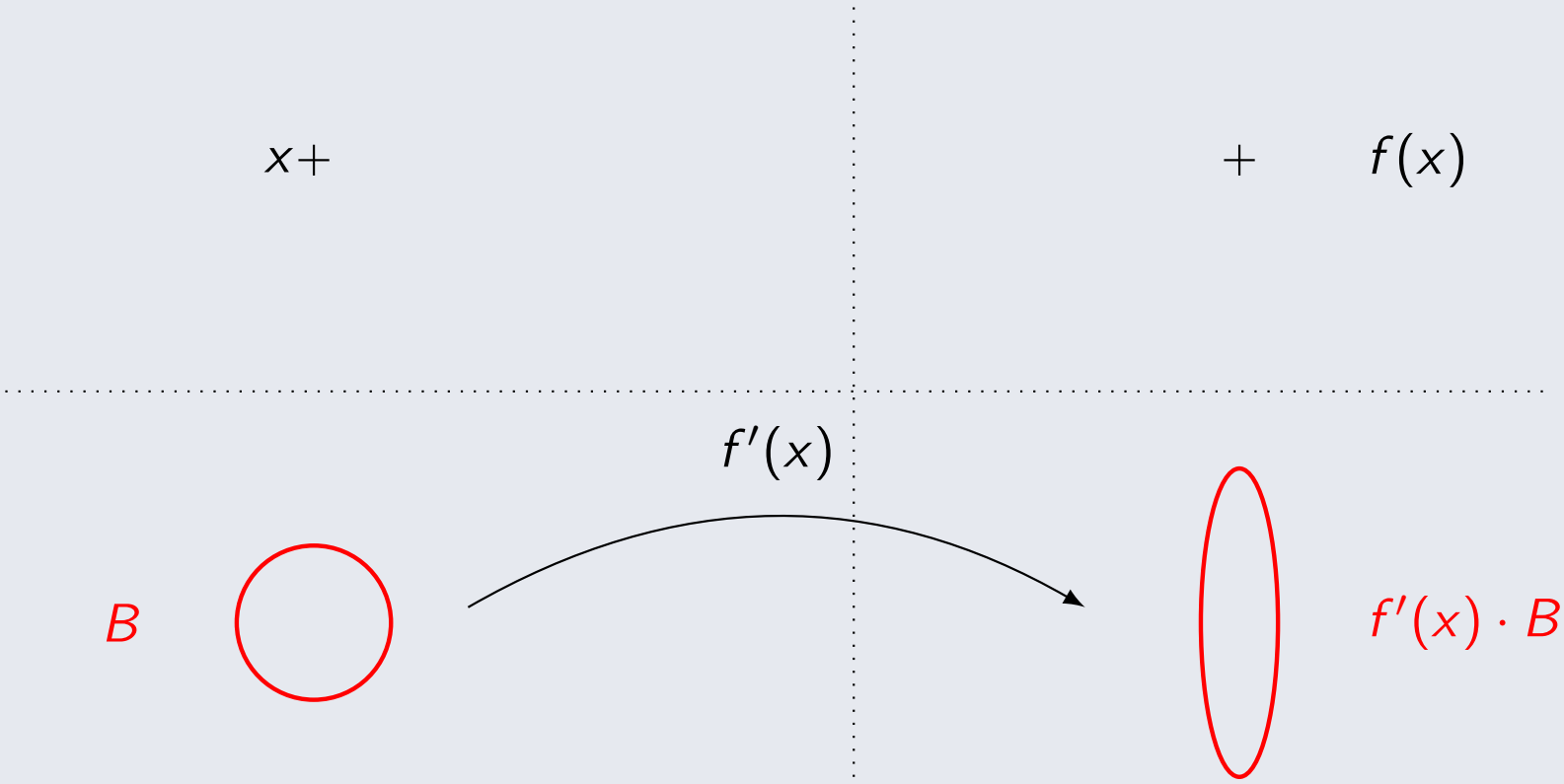

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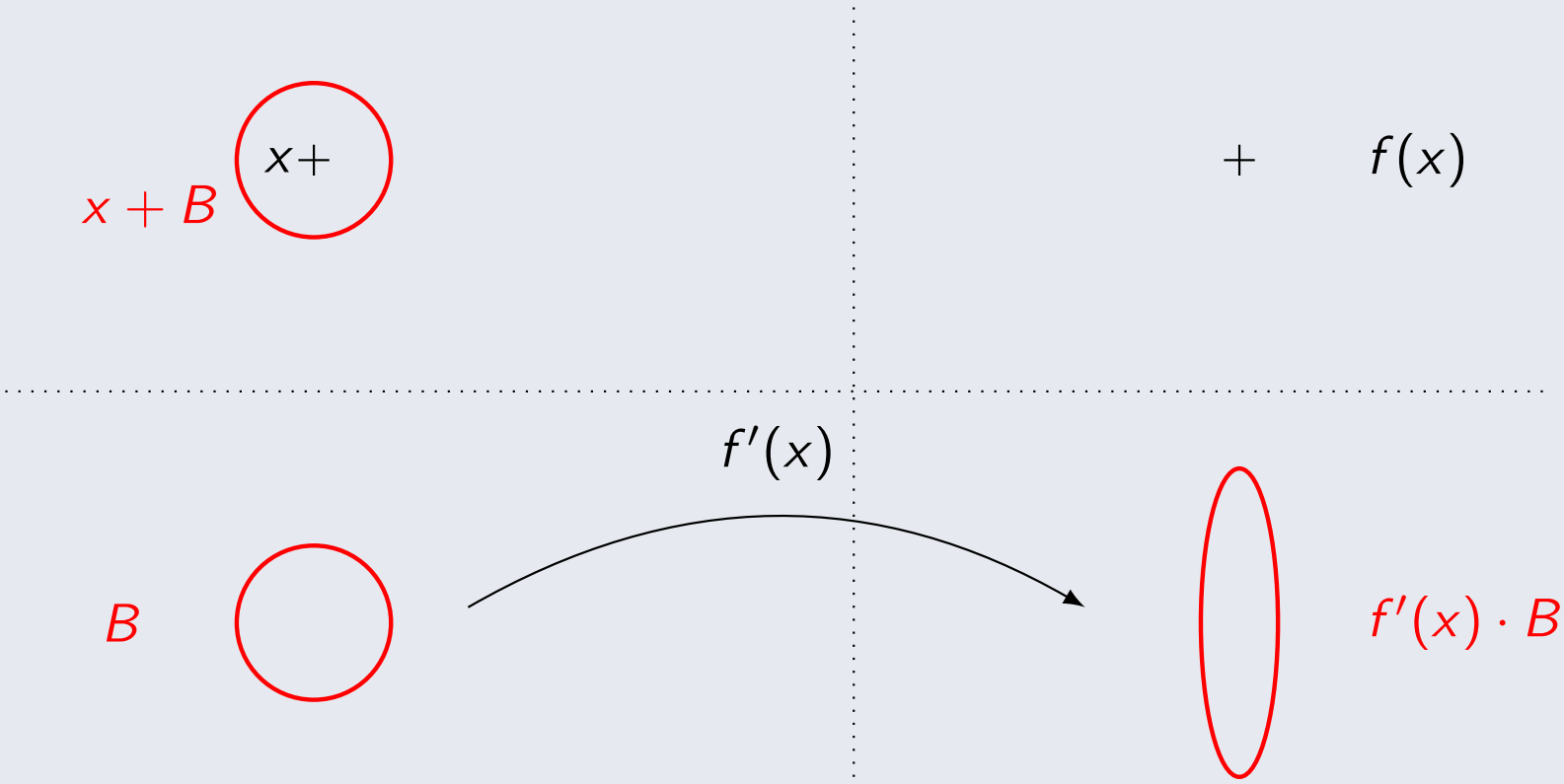

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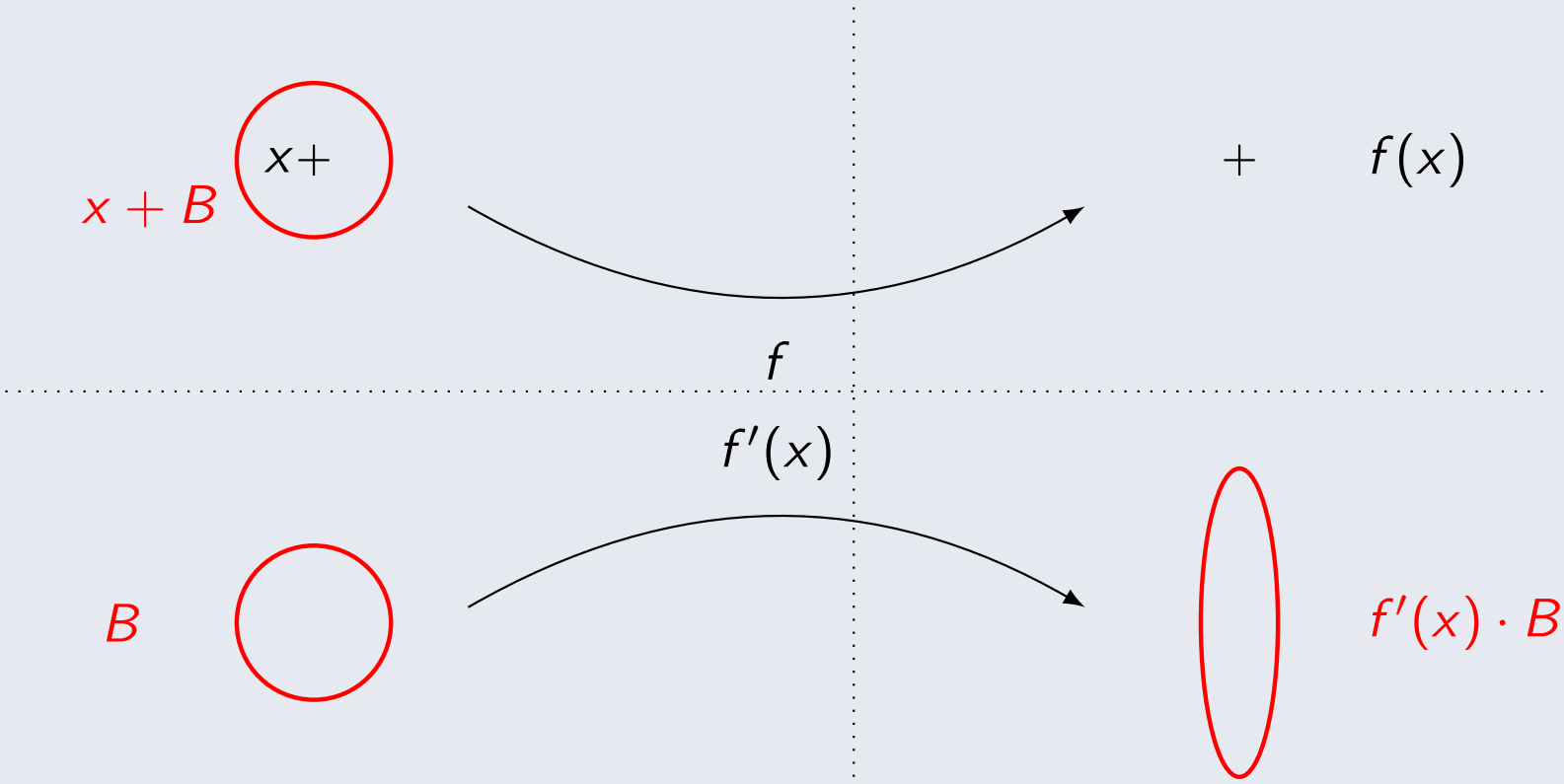

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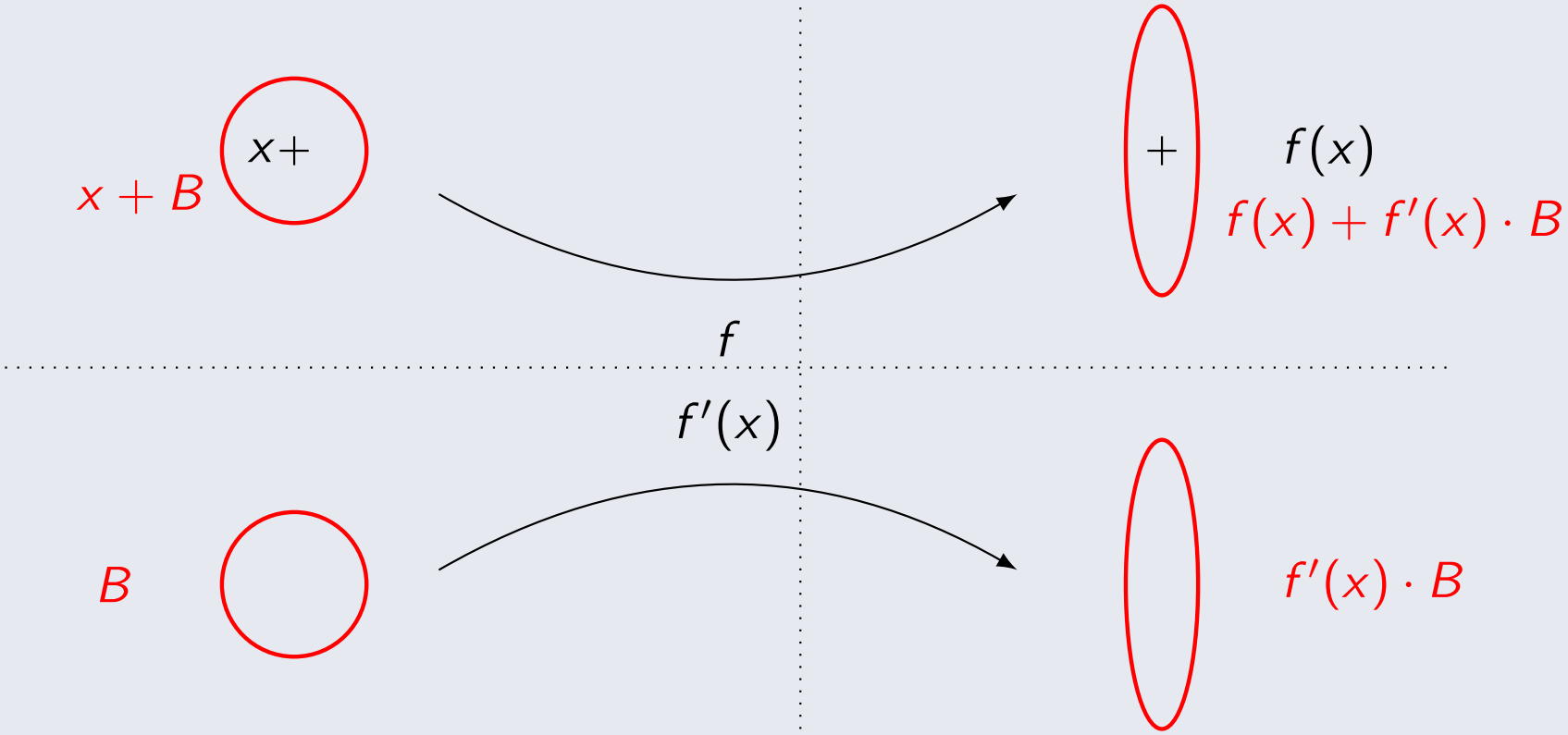

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# Introduction to the Somos-4 sequence

## Definition

We define the **Somos-4** sequence by recursion, with :

$$x_0, x_1, x_2, x_3 \in \mathbb{Z}_p^\times,$$

$$x_{n+4} = \frac{x_{n+1}x_{n+3} + x_{n+2}^2}{x_n}.$$

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$$x_0, x_1, x_2, x_3 \in \mathbb{Z}_p^\times,$$

$$x_{n+4} = \frac{x_{n+1}x_{n+3} + x_{n+2}^2}{x_n}.$$

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This formula comes from the $Z$-coordinate of $[m]\,P + Q$ for some $P, Q$ points on the **elliptic curve** $y^2 + y = x^3 + x$.

# Introduction to the Somos-4 sequence

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## Proposition

*For all $n$, $x_n \in \mathbb{Z}_p$, i.e. $v_p(x_n) \geq 0$.*

# The Laurent phenomenon

## Remark

If $x_0, x_1, x_2, x_3$ are known up to $O(p^m)$, then because of the division by $x_n$, a naive step-by-step analysis show that $x_{n+4}$ is known up to $O(p^{m-\sum_{k=0}^{n} v_p(x_k)})$.

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*Let $P_n$ be the rational fraction defined by the recursion formula defining Somos-4 :*

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If $m$ is big enough,

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## Corollary

*There is no intrinsic loss of precision : $x_n$ is determined up to $O(p^m)$.*

# Table of contents

# Lattices

# Lattices

## Lemma

*Let $f : \mathbb{Q}_p^n \to \mathbb{Q}_p^m$ be a **differentiable** mapping.*
*Let $x \in \mathbb{Q}_p^n$. We assume that $f'(x)$ is **surjective**.*
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## Remark

Our framework can be extended to (complete) **ultrametric** $K$-vector spaces (*e.g.* $\mathbb{F}_p((X))^n$, $\mathbb{Q}((X))^m$).

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As a consequence,

$$\frac{1}{k!} f^{(k)}(x) \cdot (p^m \mathbb{Z}_p)^k \subset p^m \mathbb{Z}_p.$$

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# Computation in SOMOS-4

## Loss in precision in SOMOS-4 with Sage ?!

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## An explanation

The **gain** in precision in $x_8$ is invisible.

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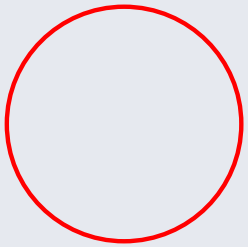
# More on the differential method

## Differential tracking of precision

$x + O(p^N)$

$x+$

$B$



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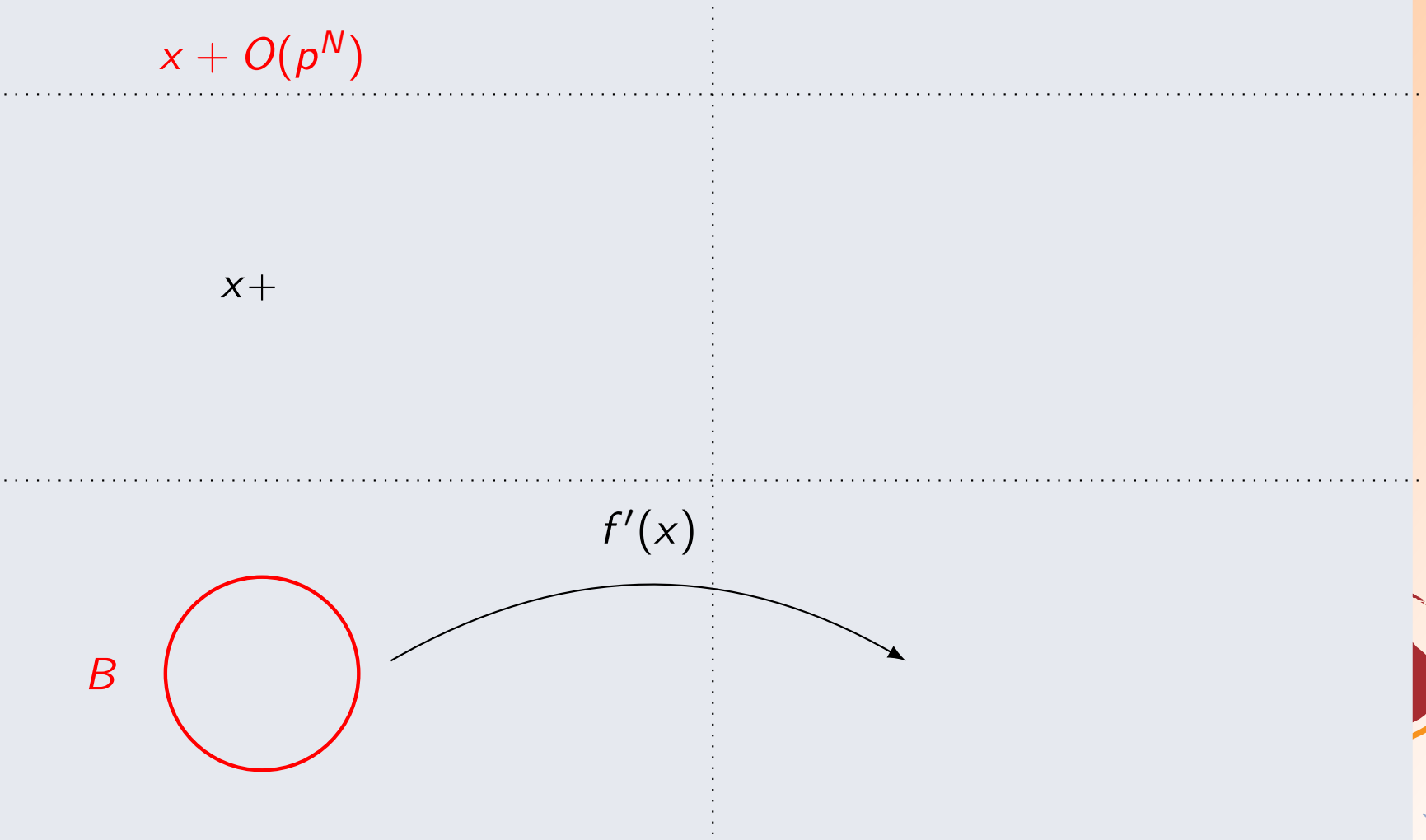

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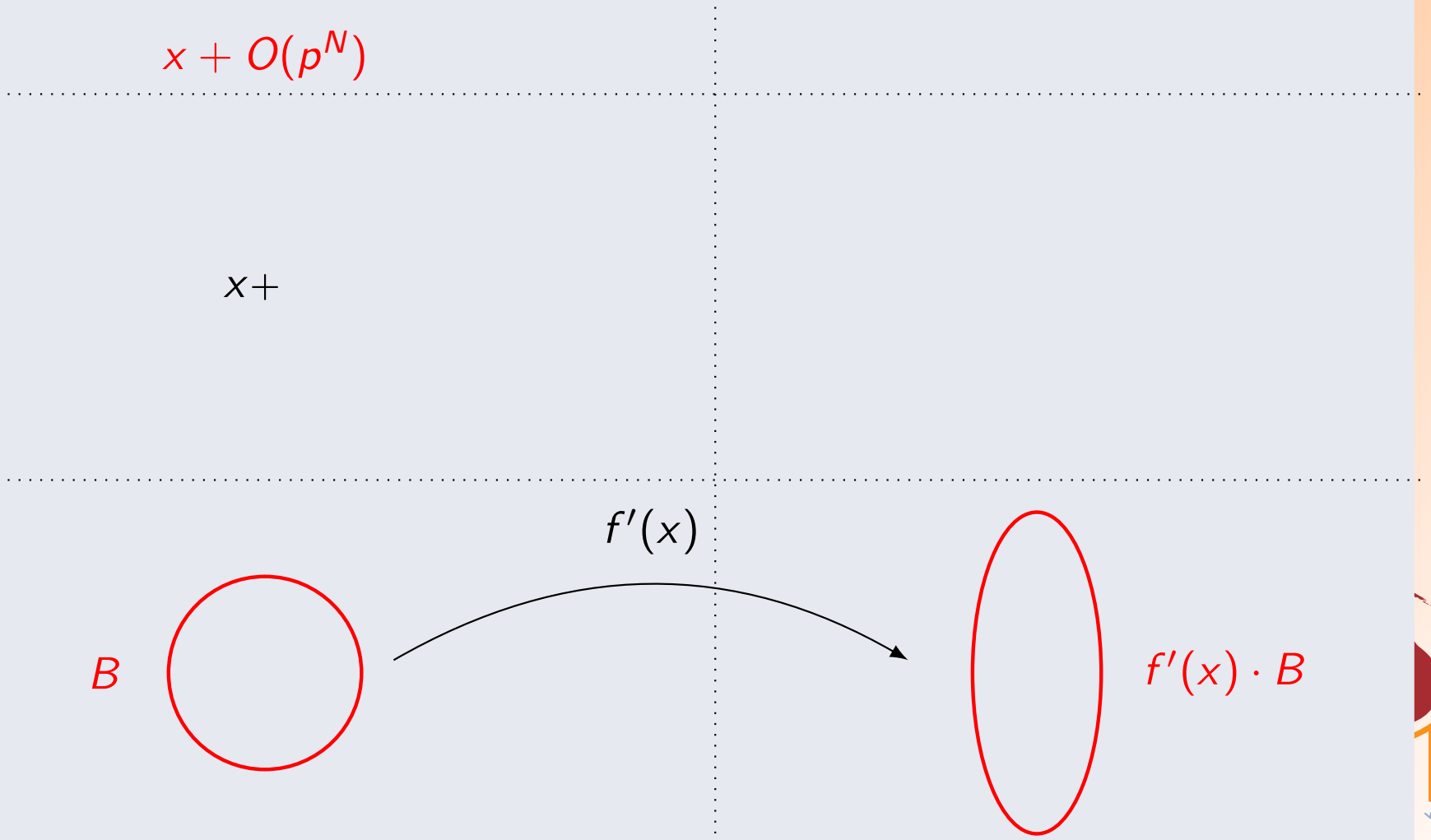

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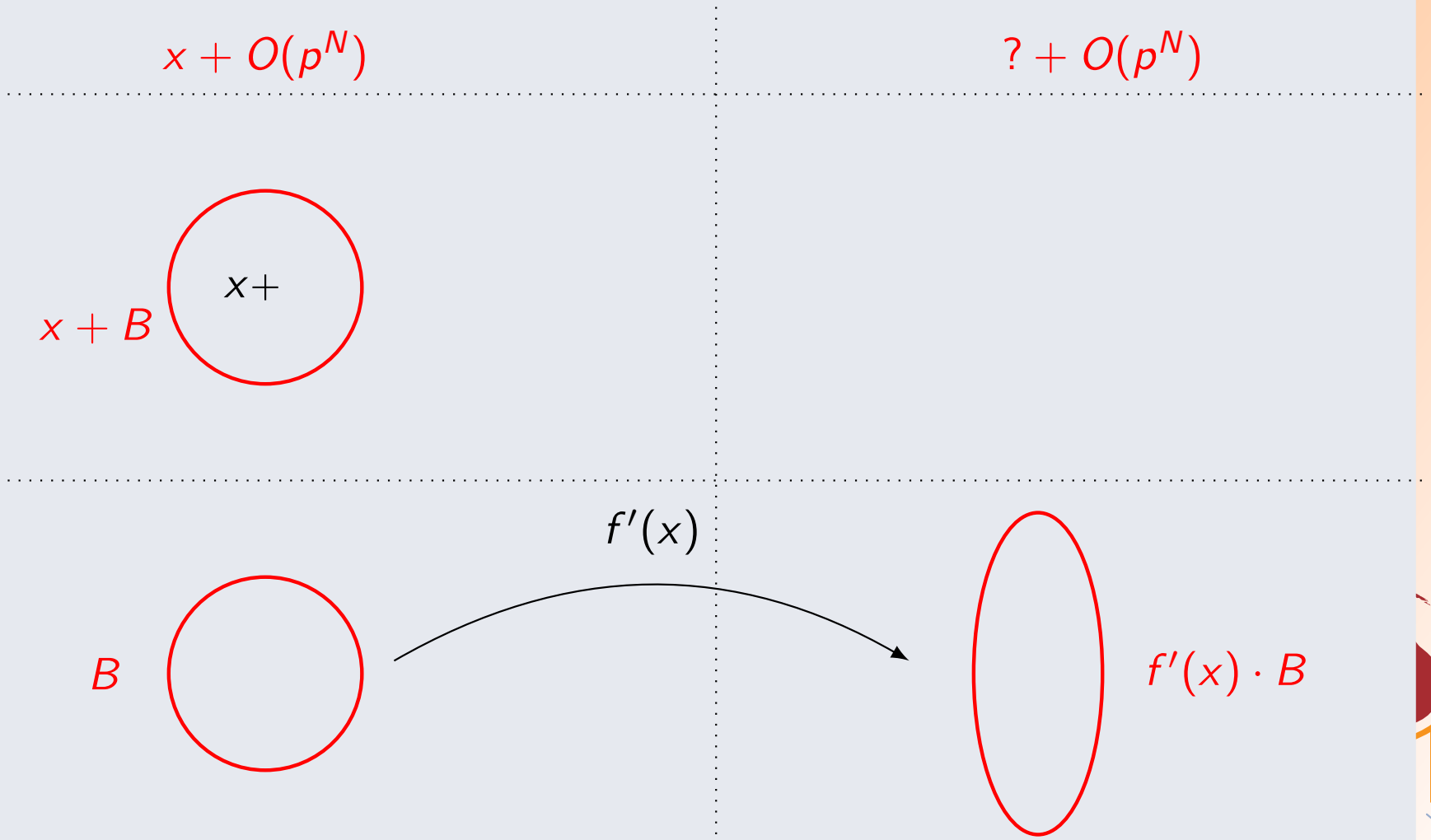

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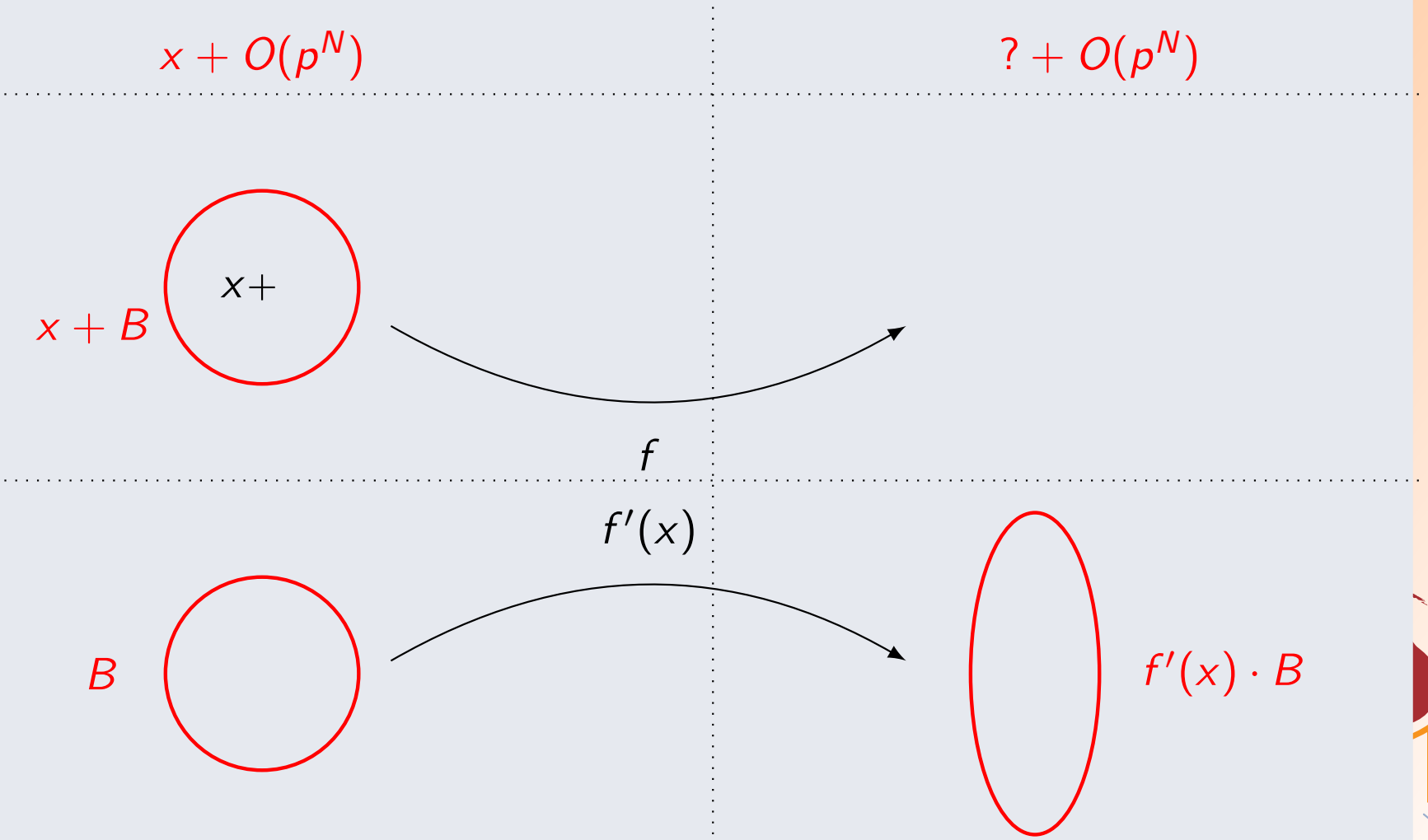

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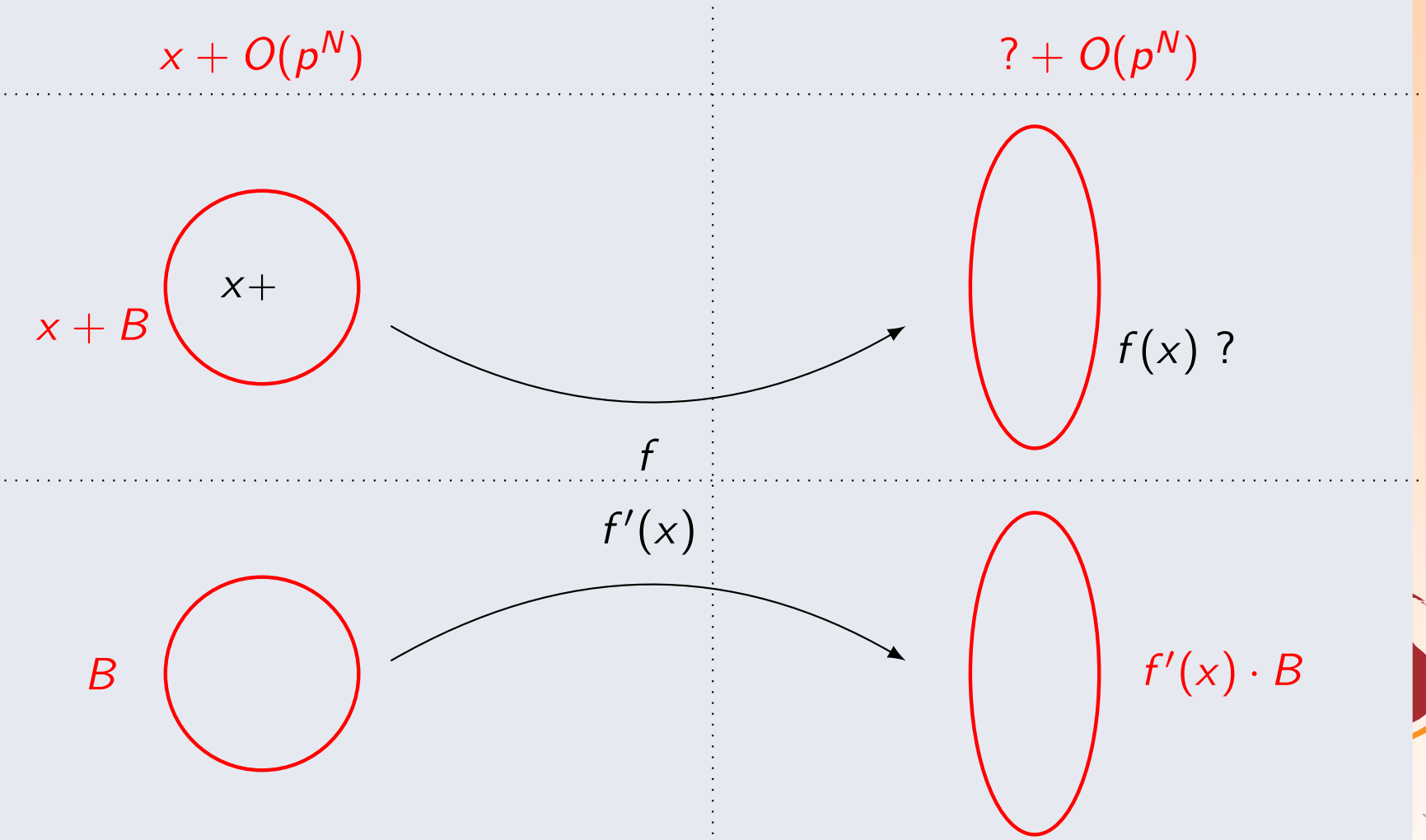

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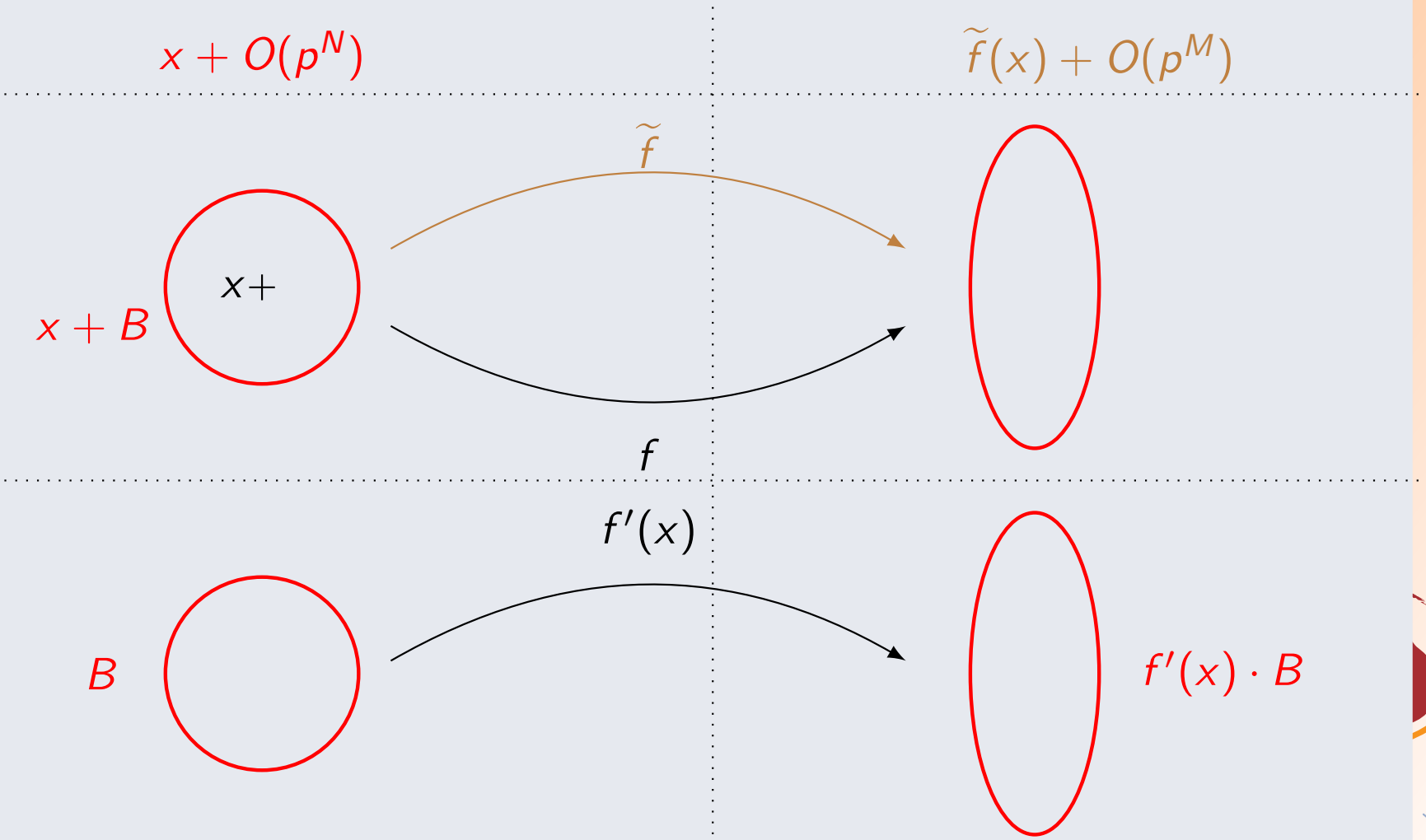

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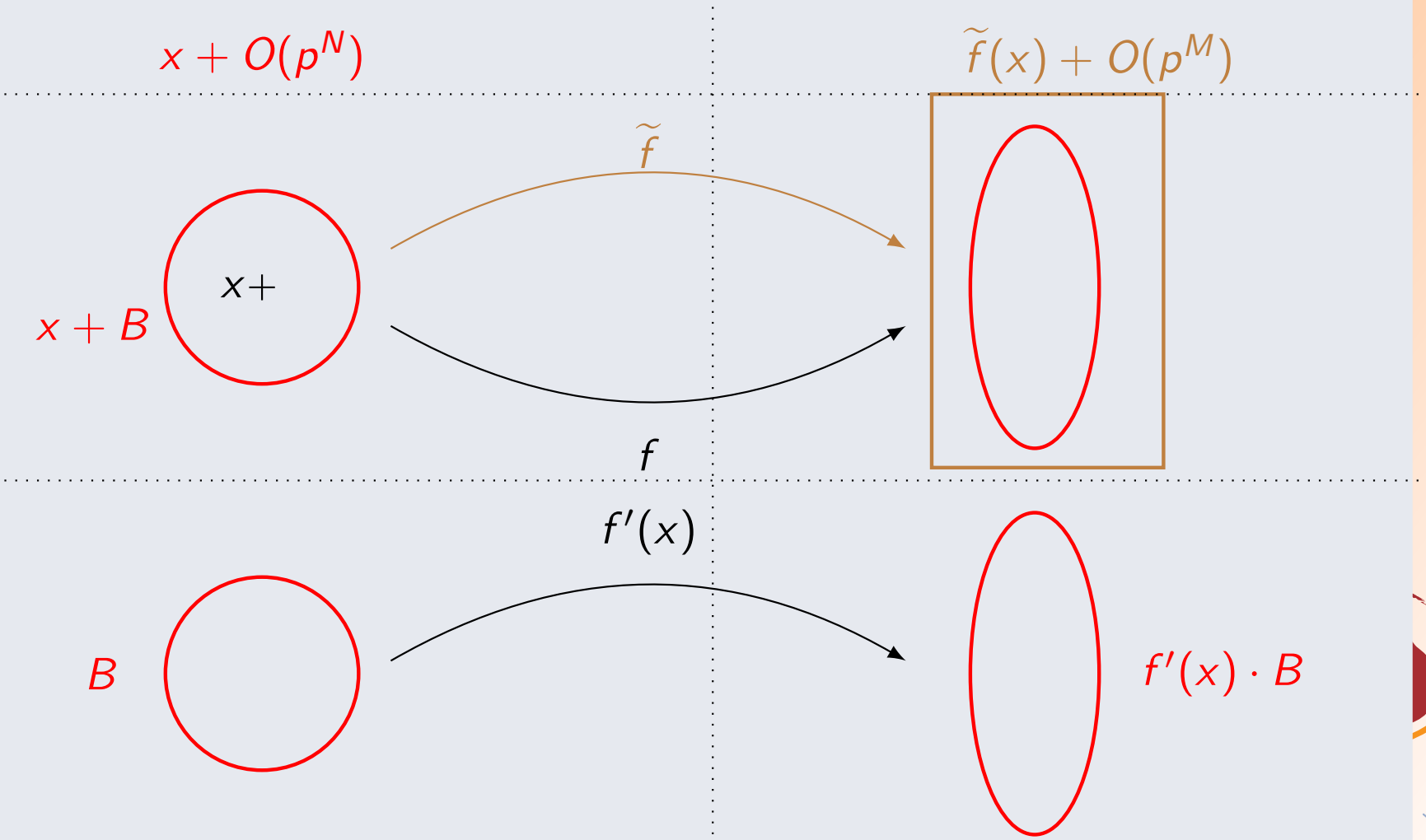

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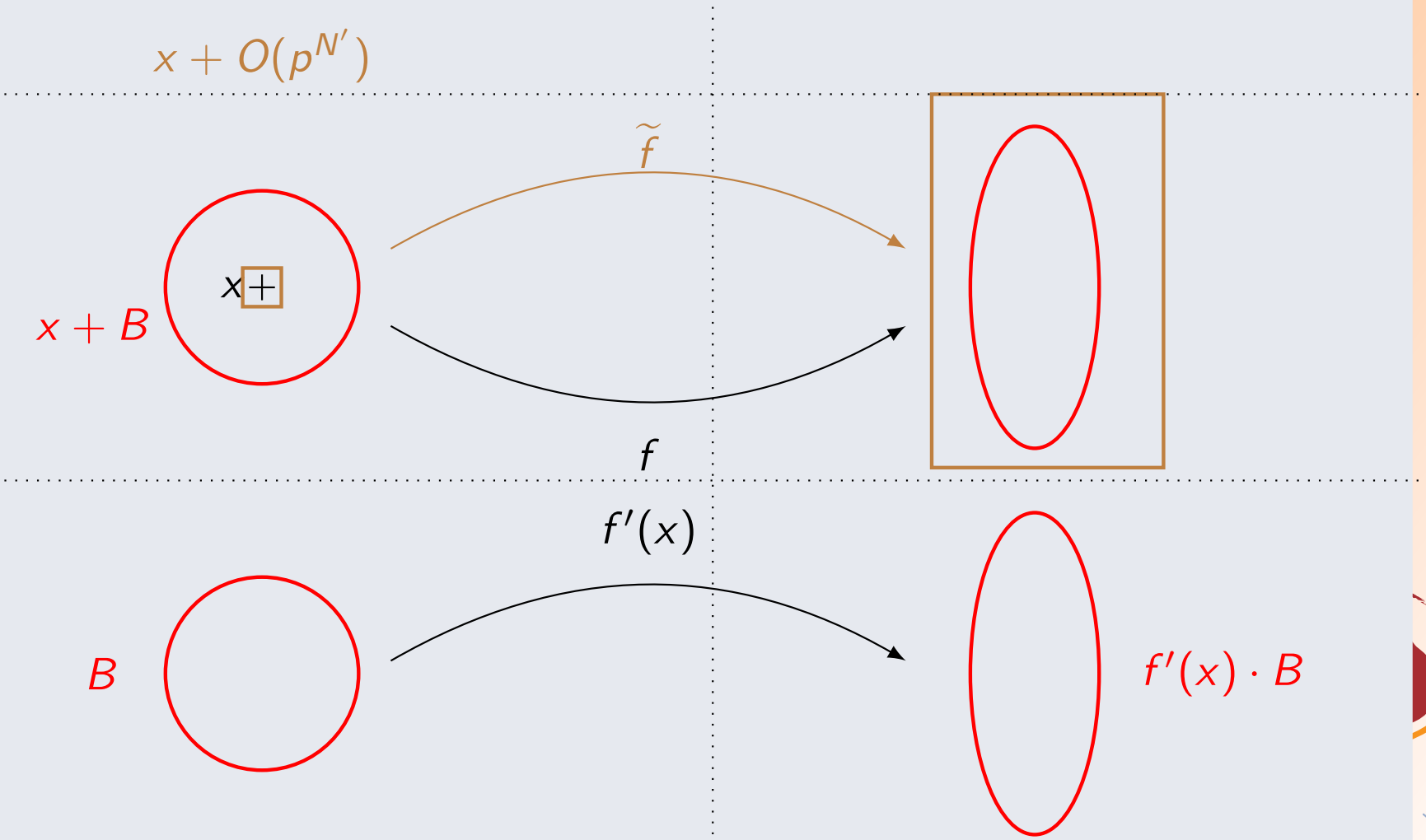

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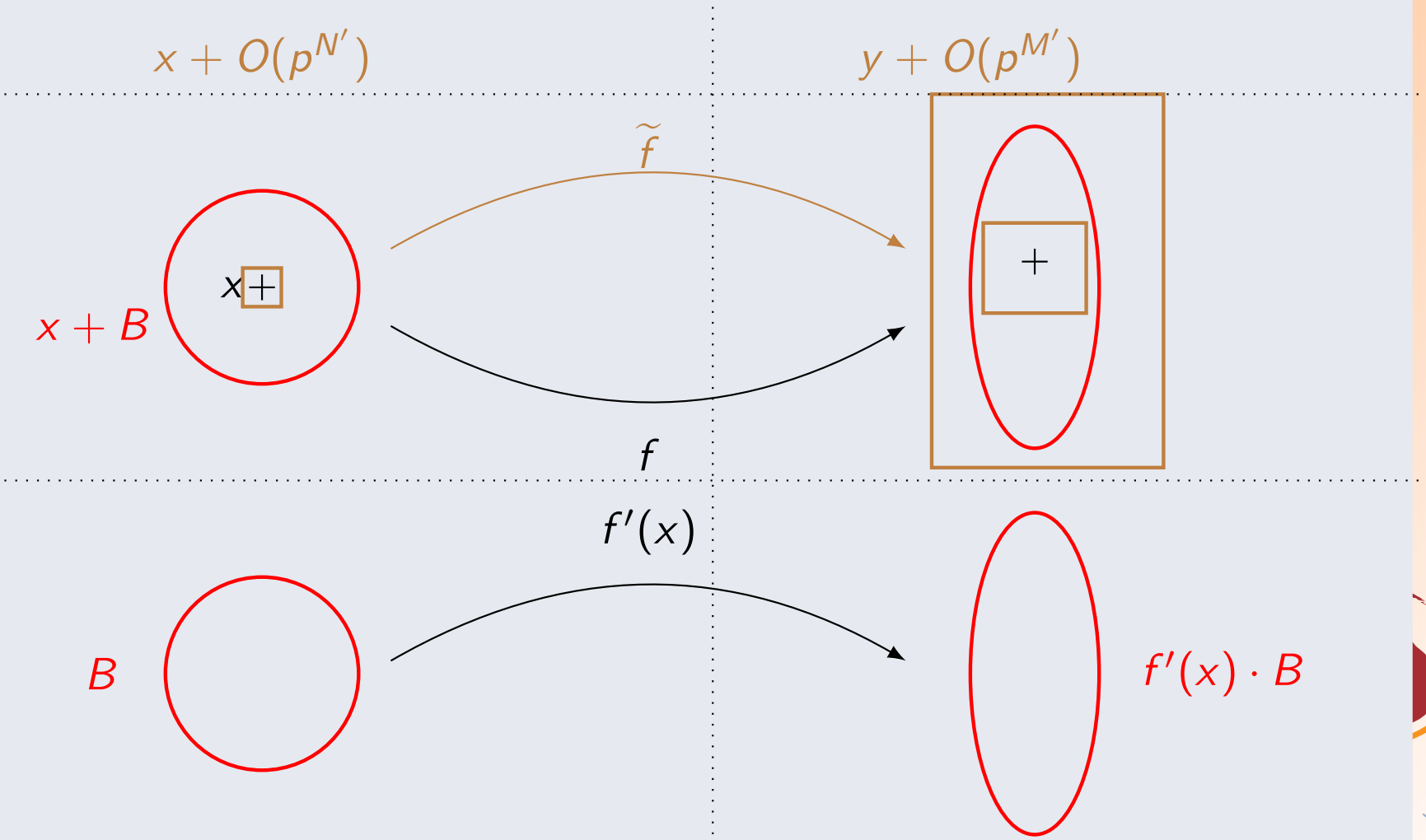

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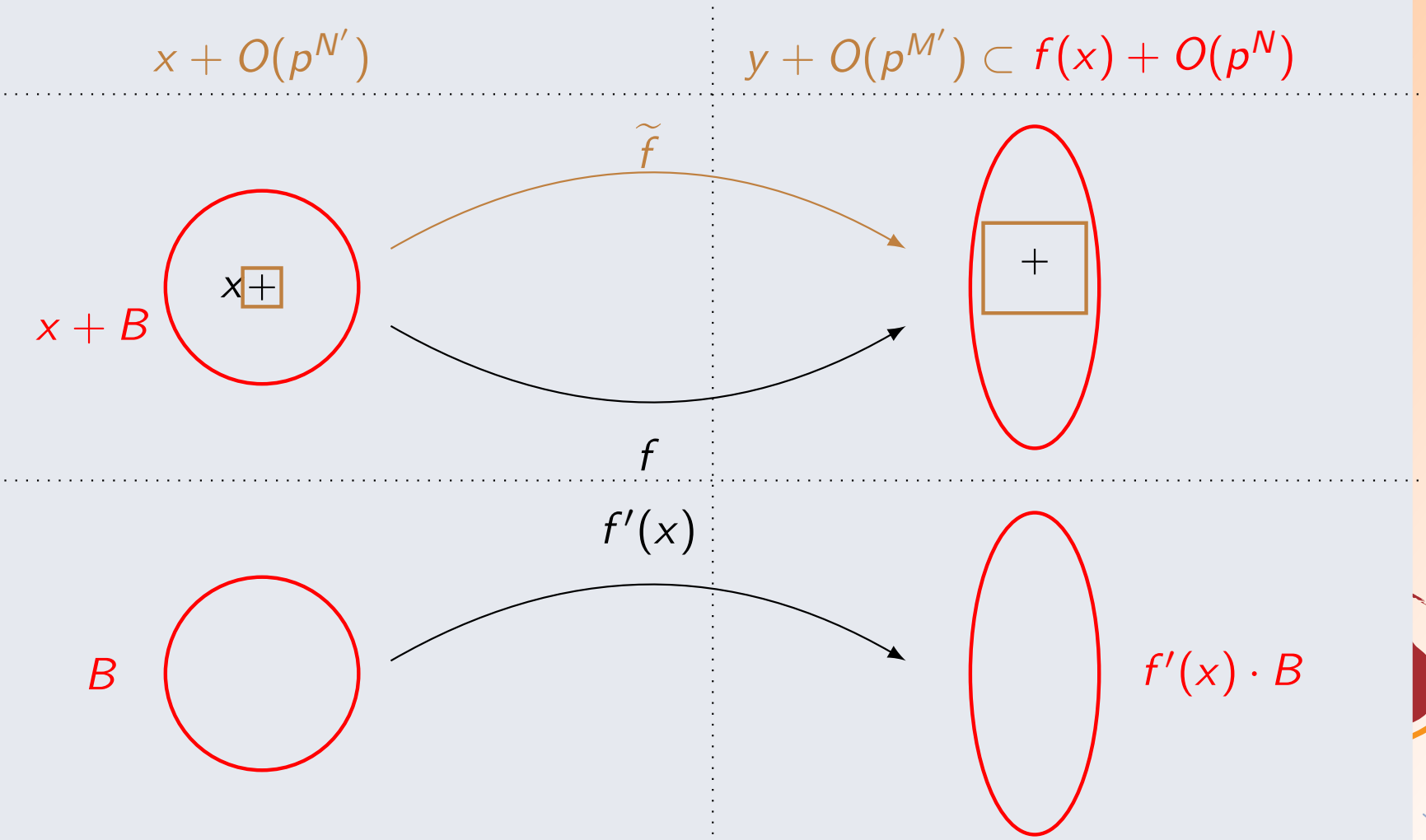

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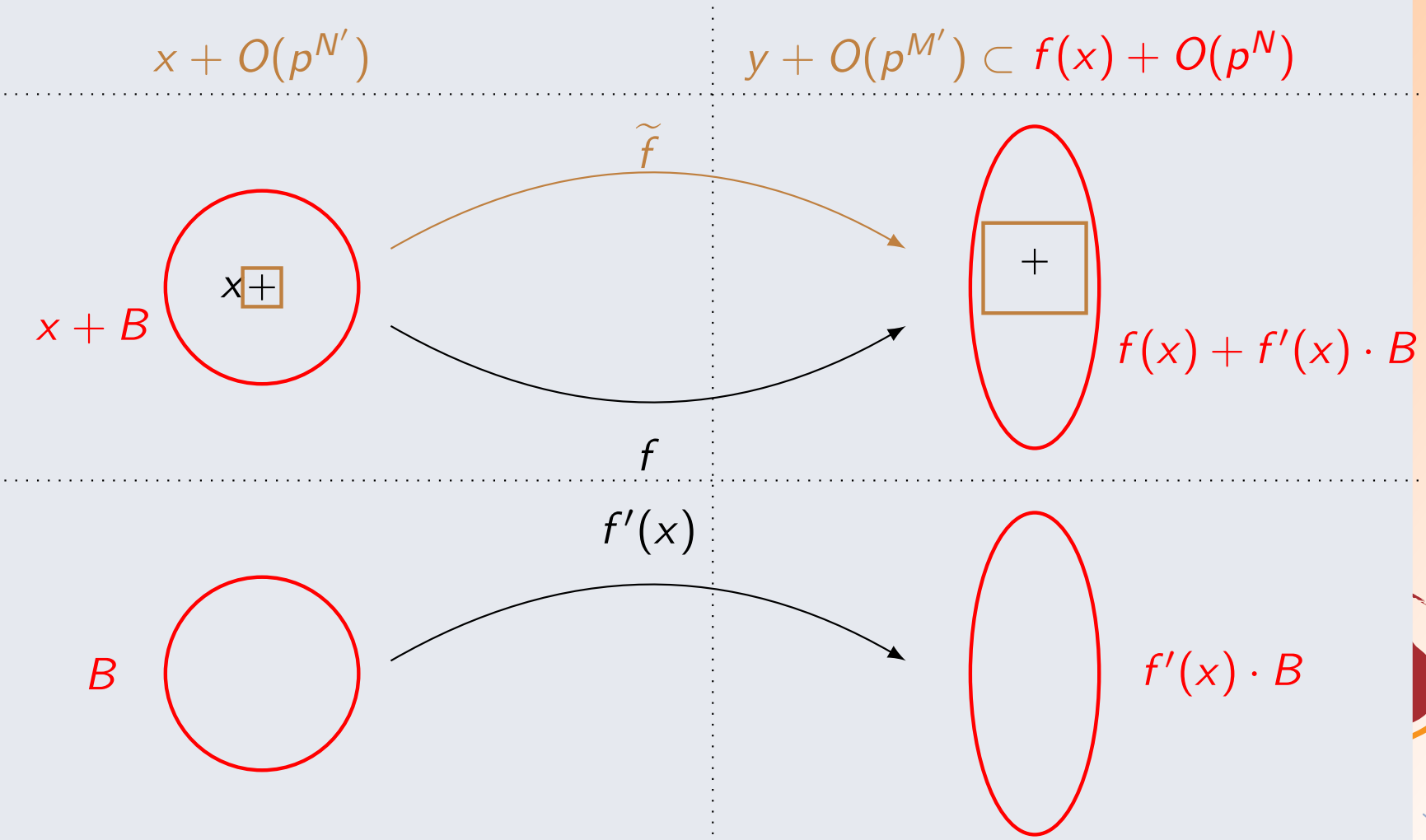

# Lifting techniques

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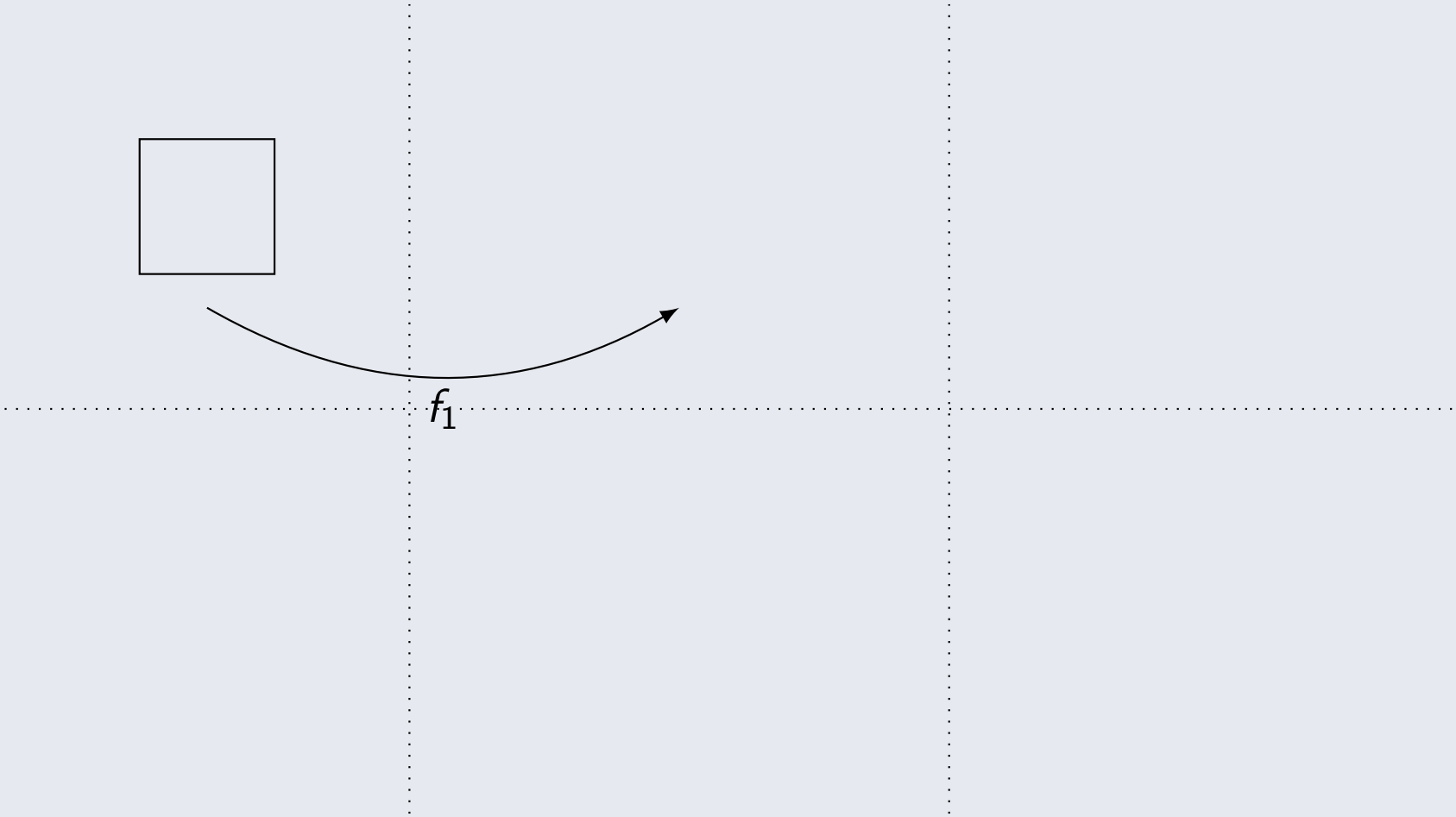

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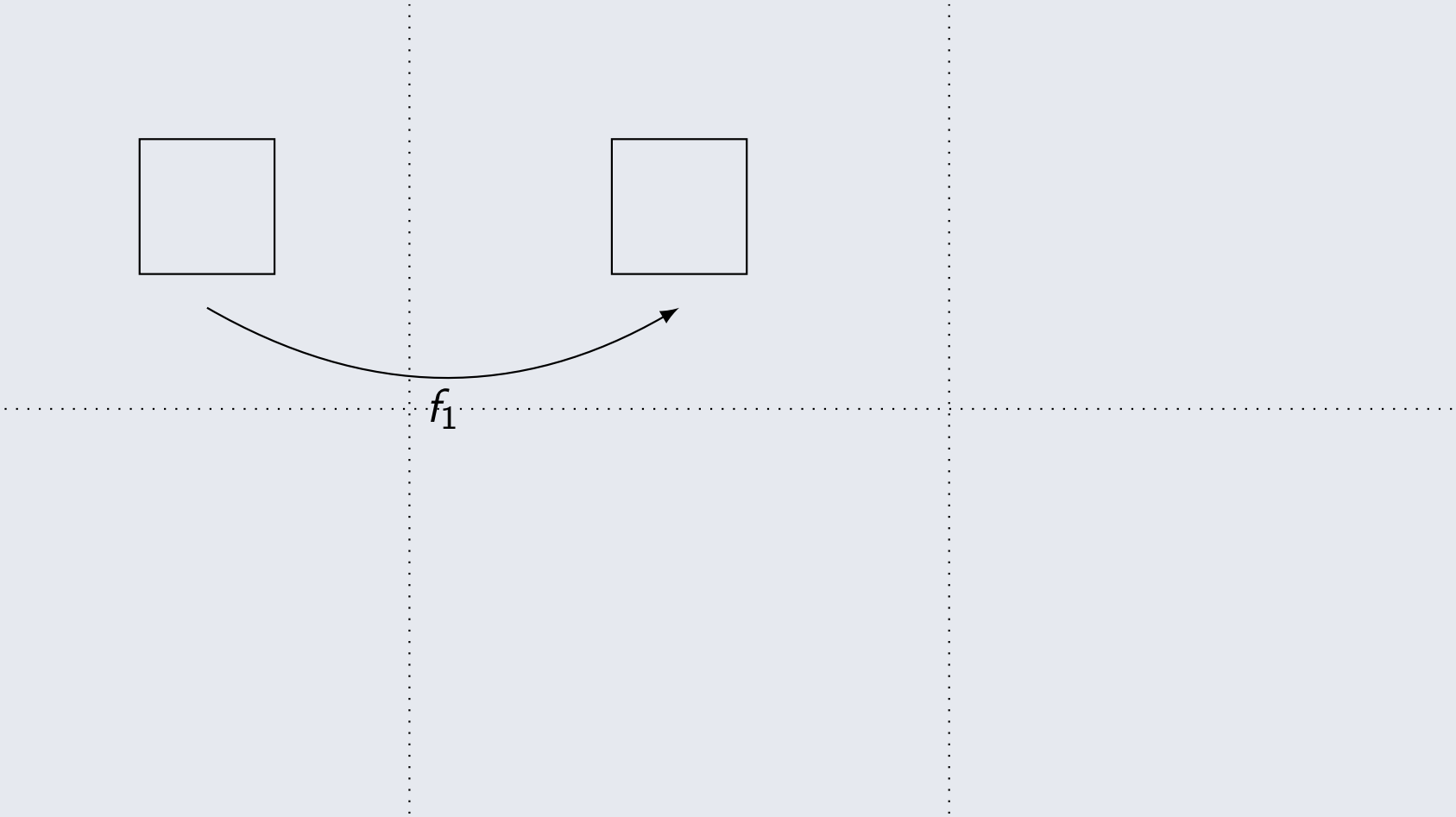

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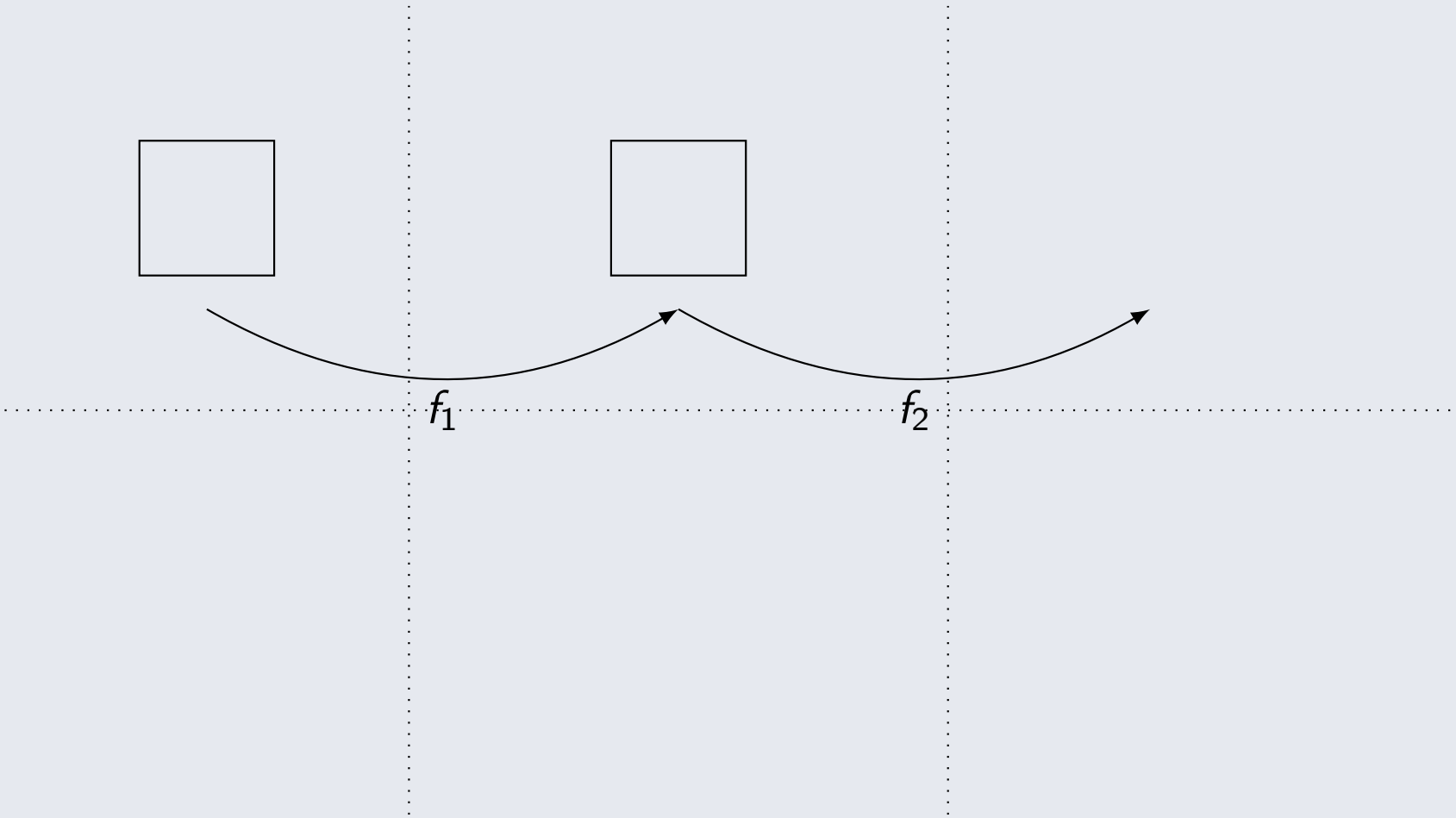

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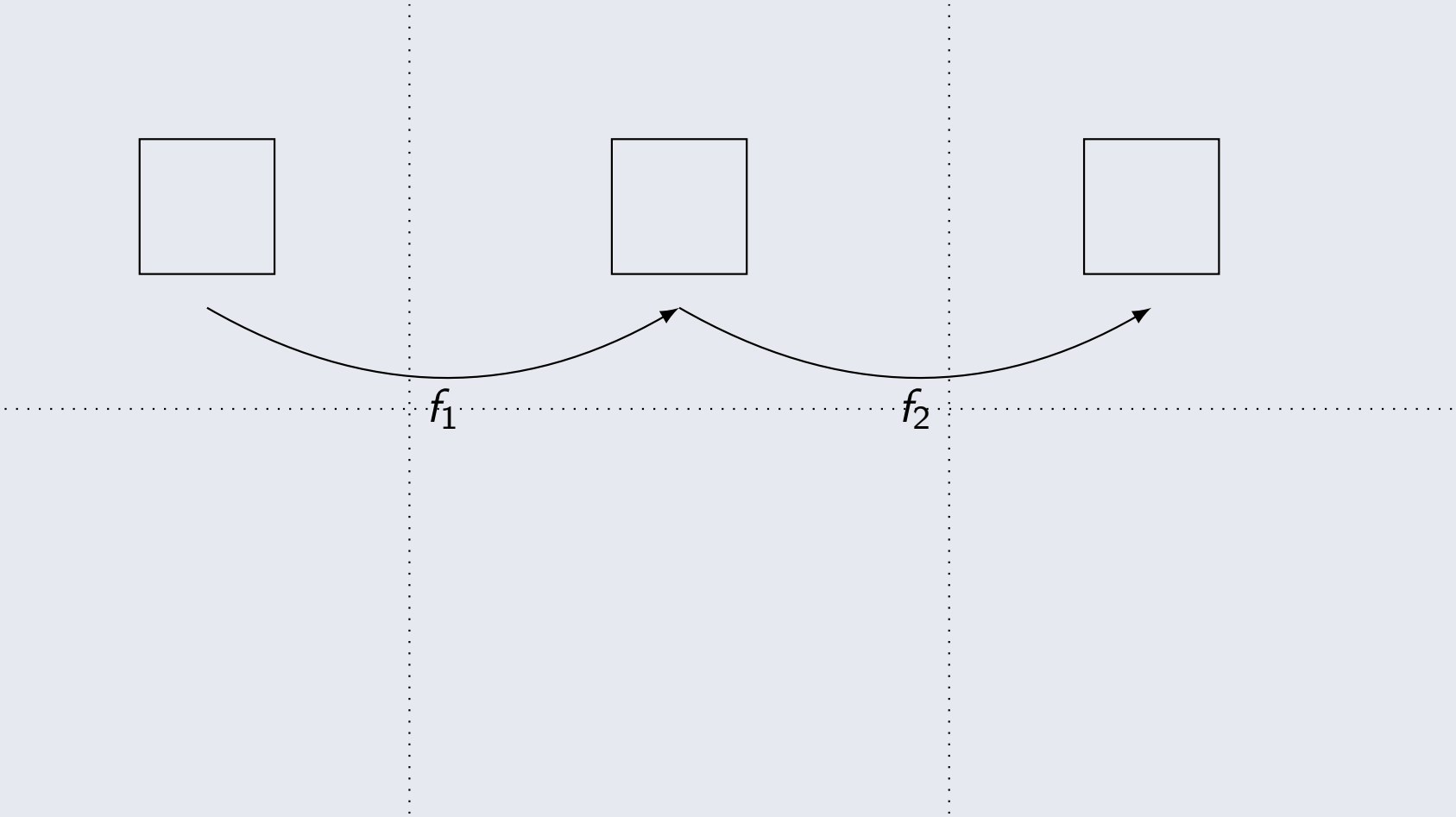

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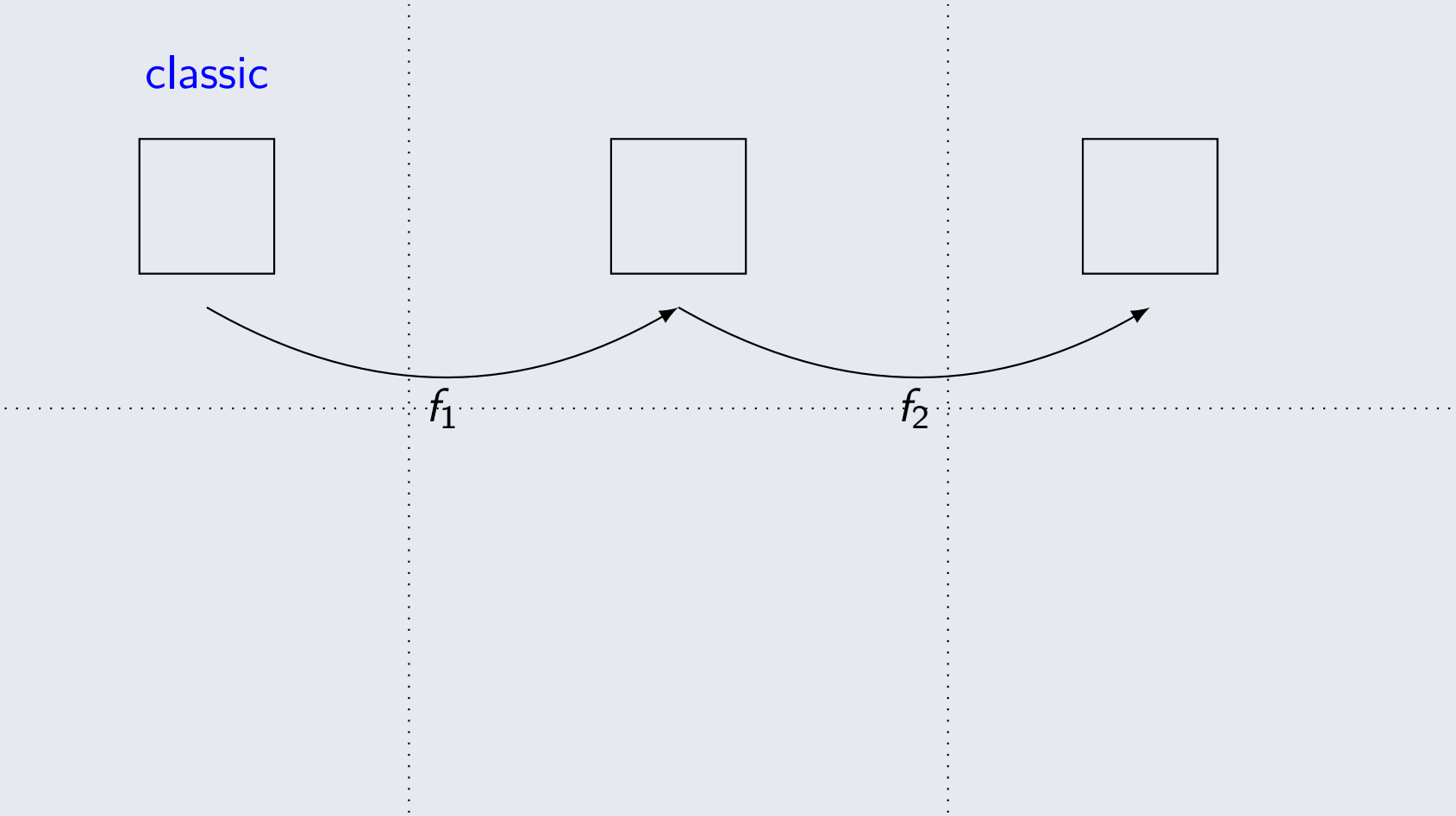

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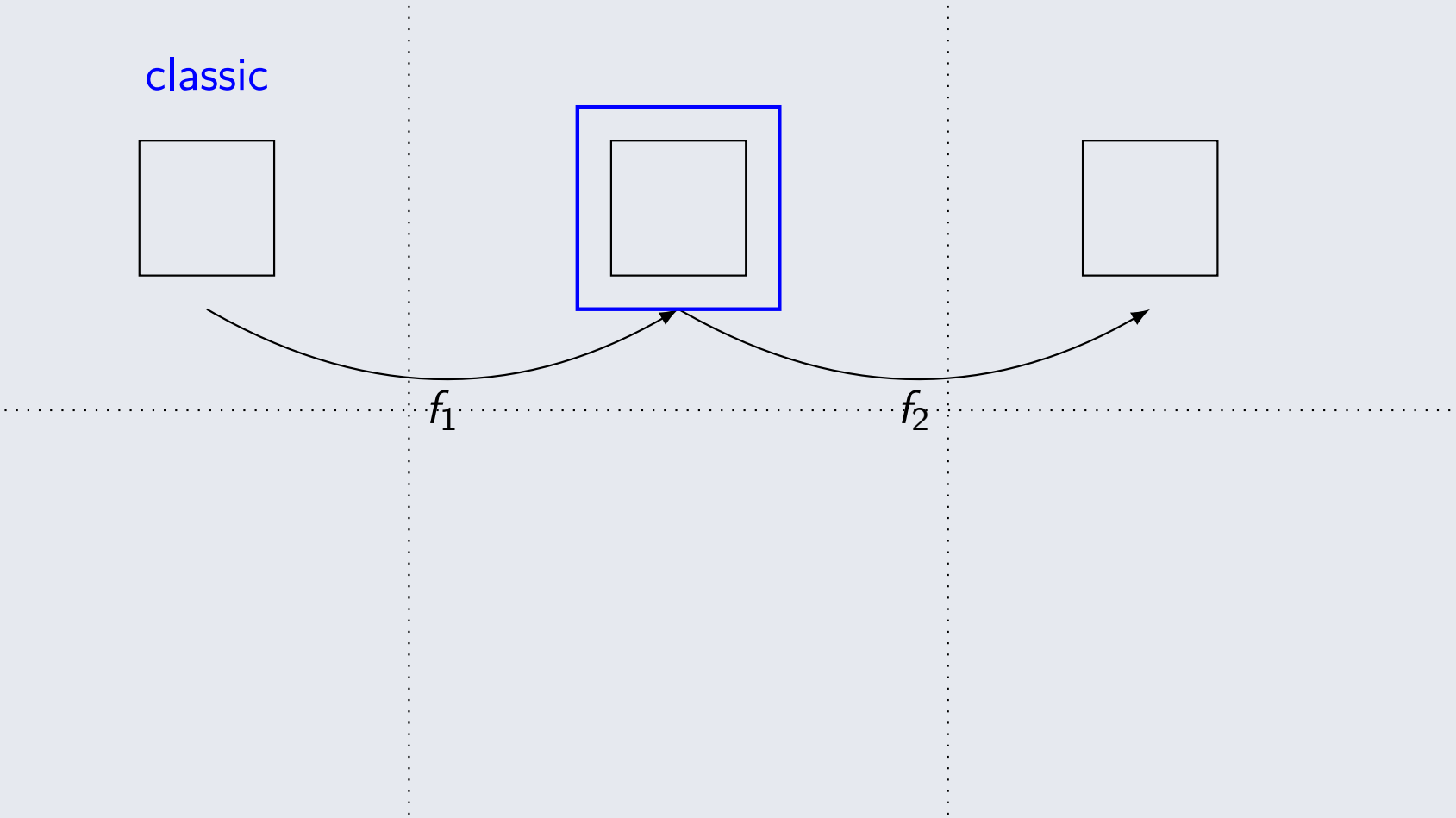

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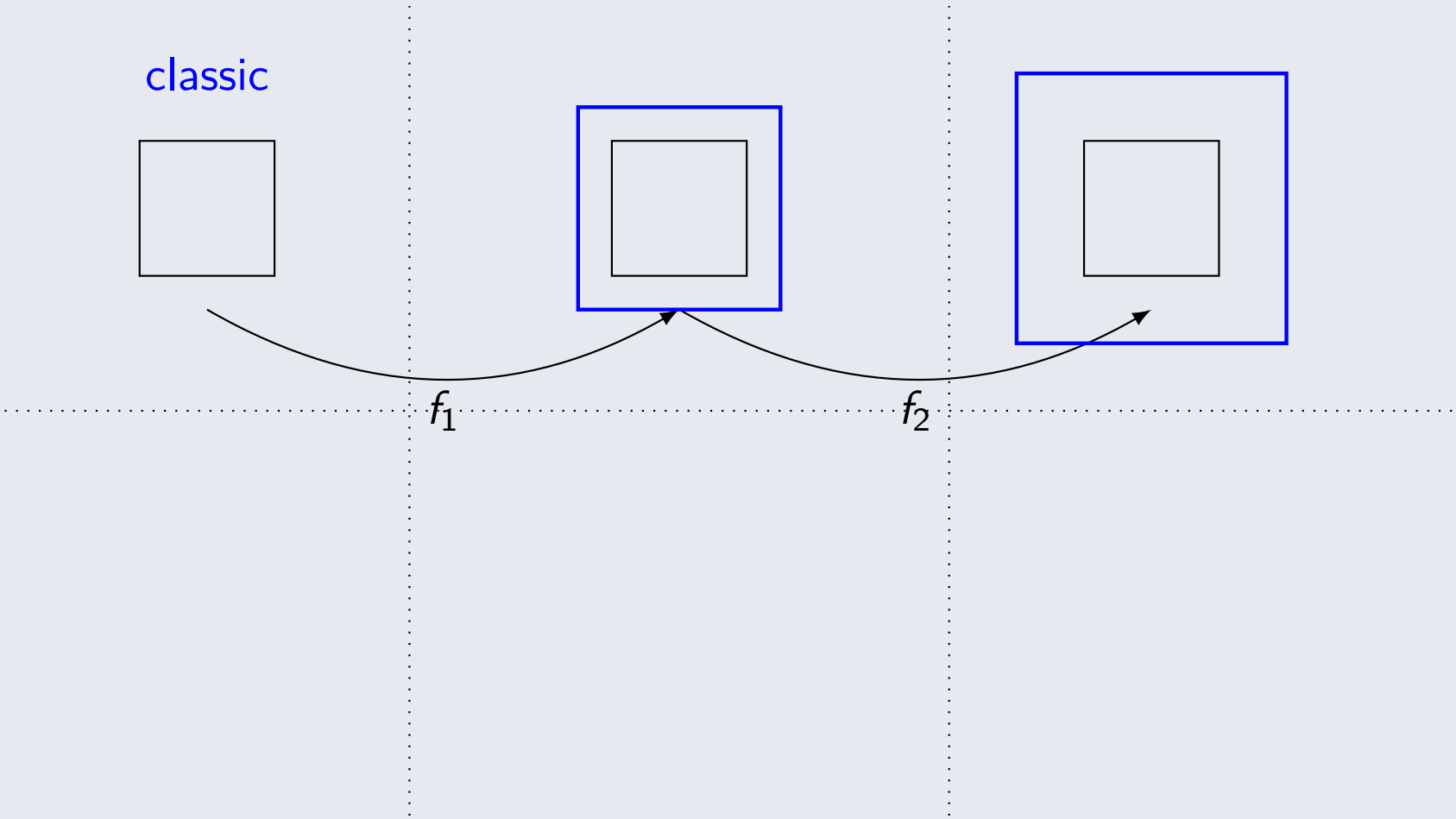

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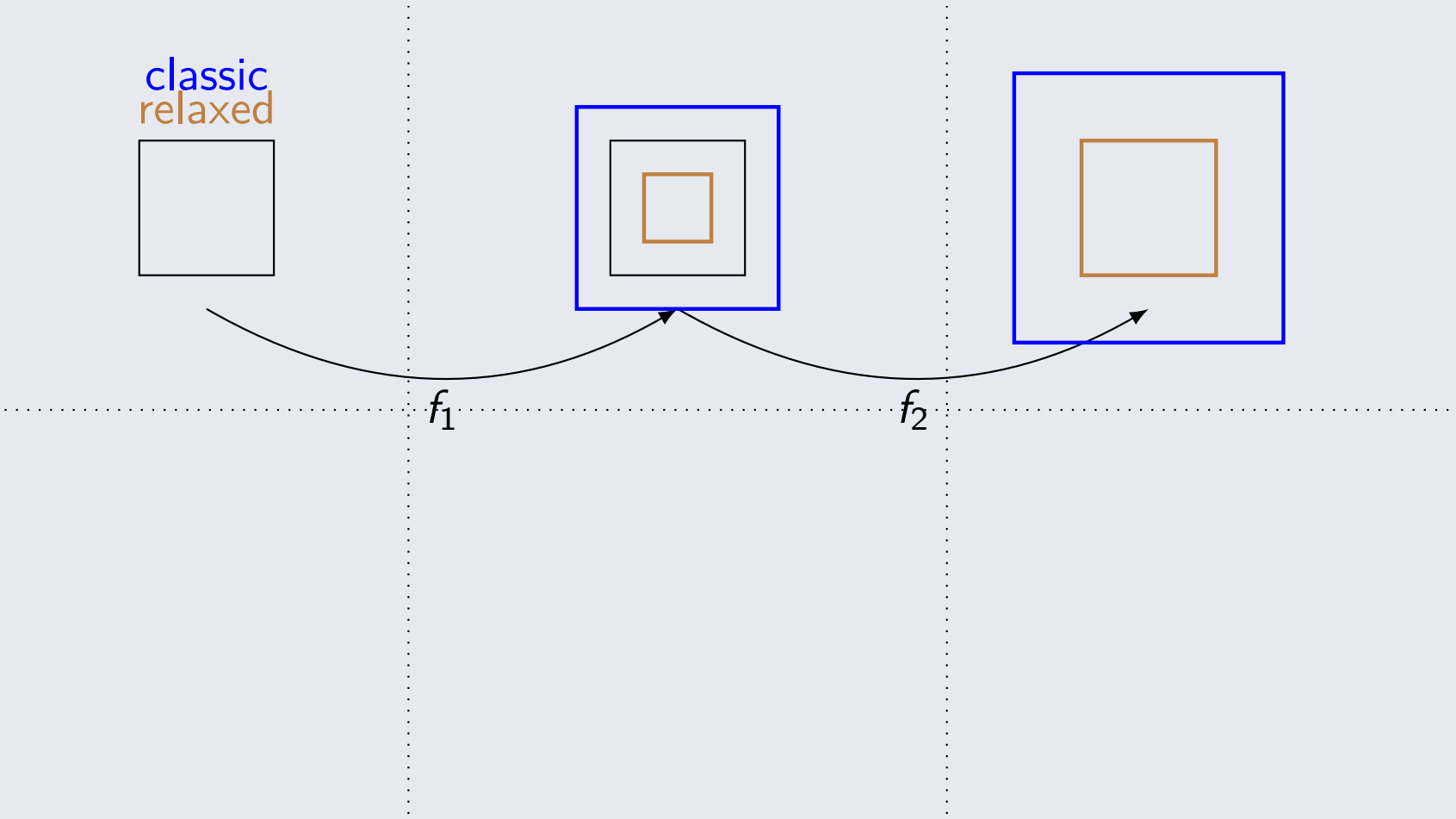

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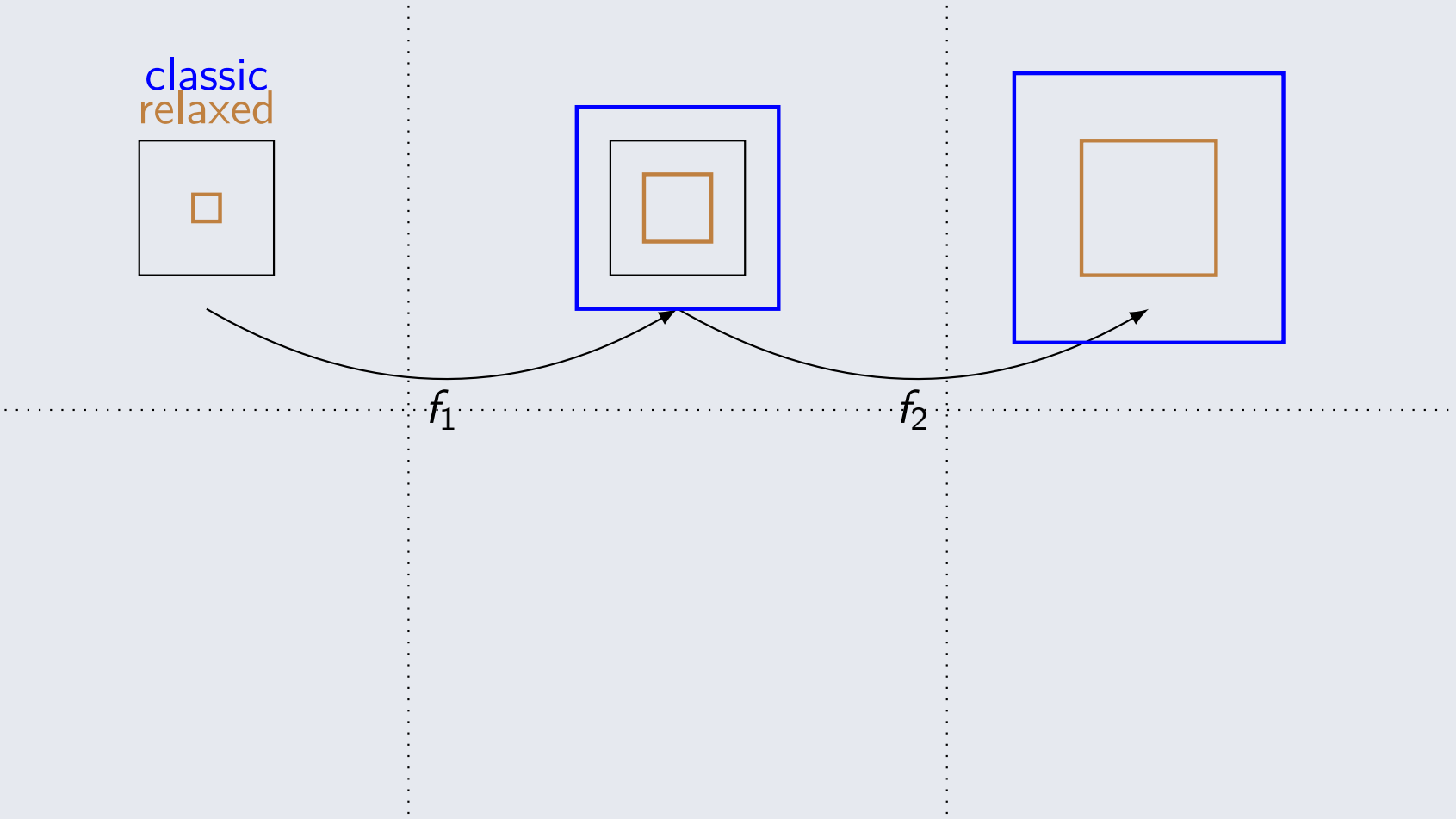

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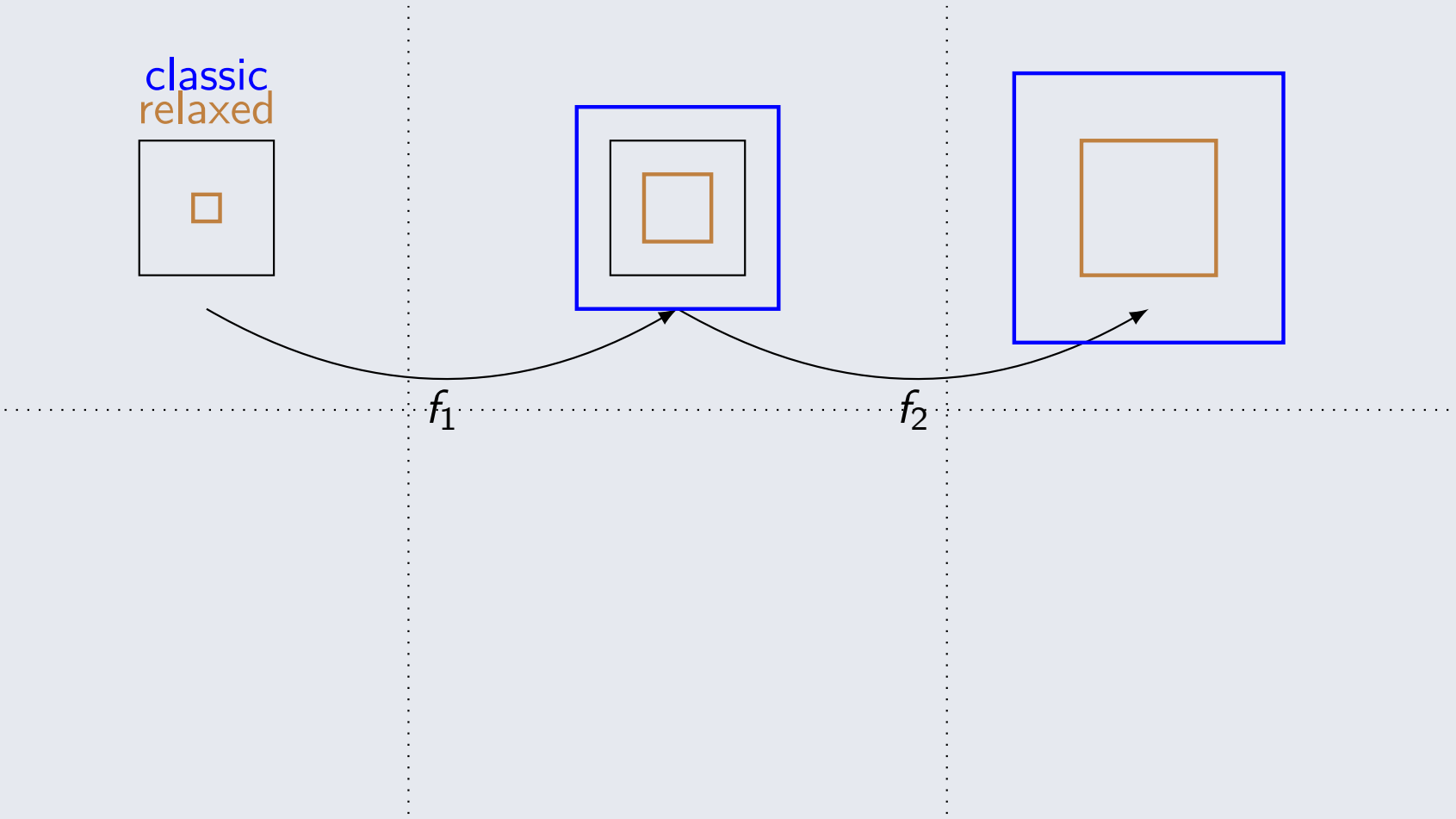

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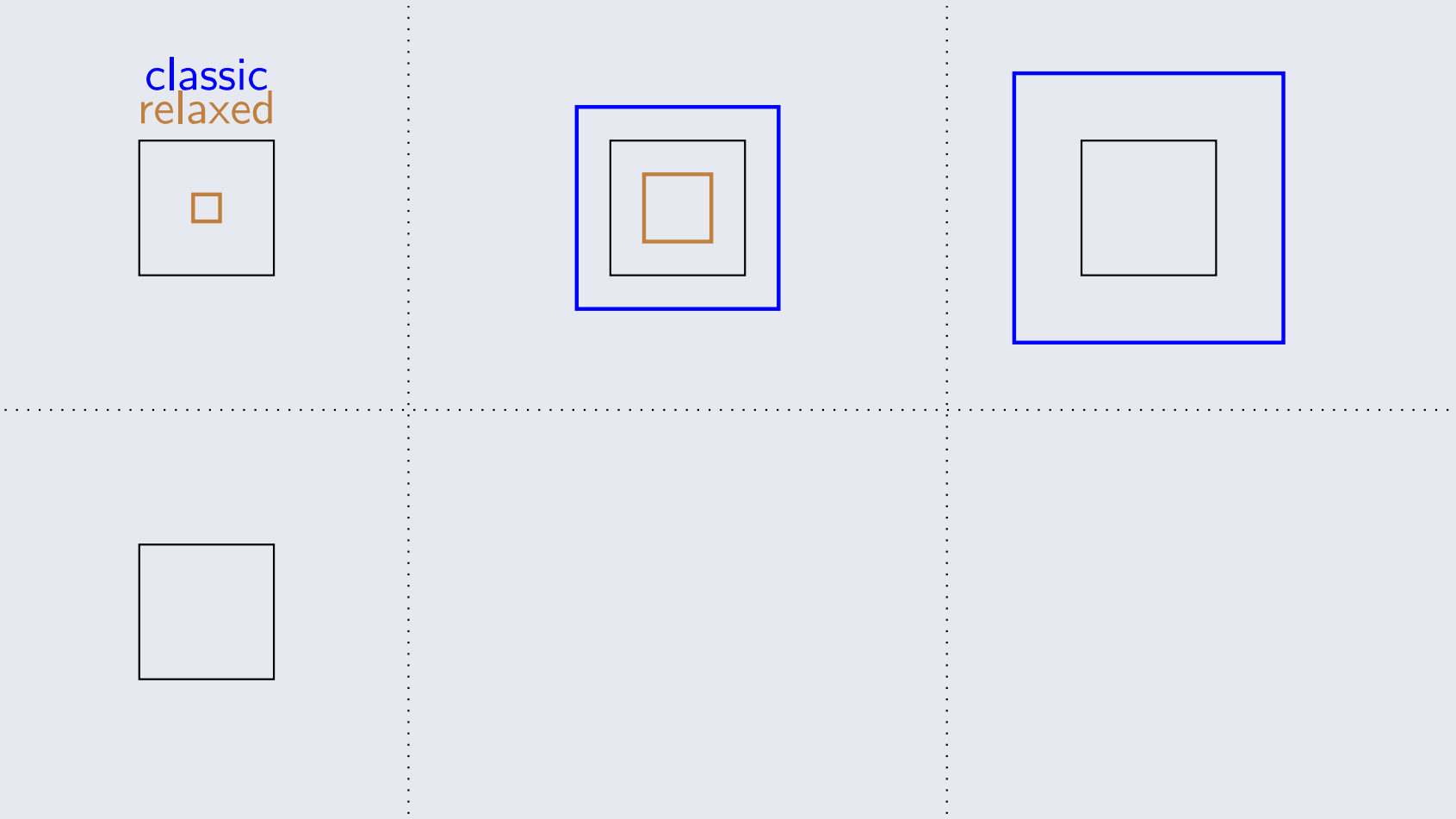

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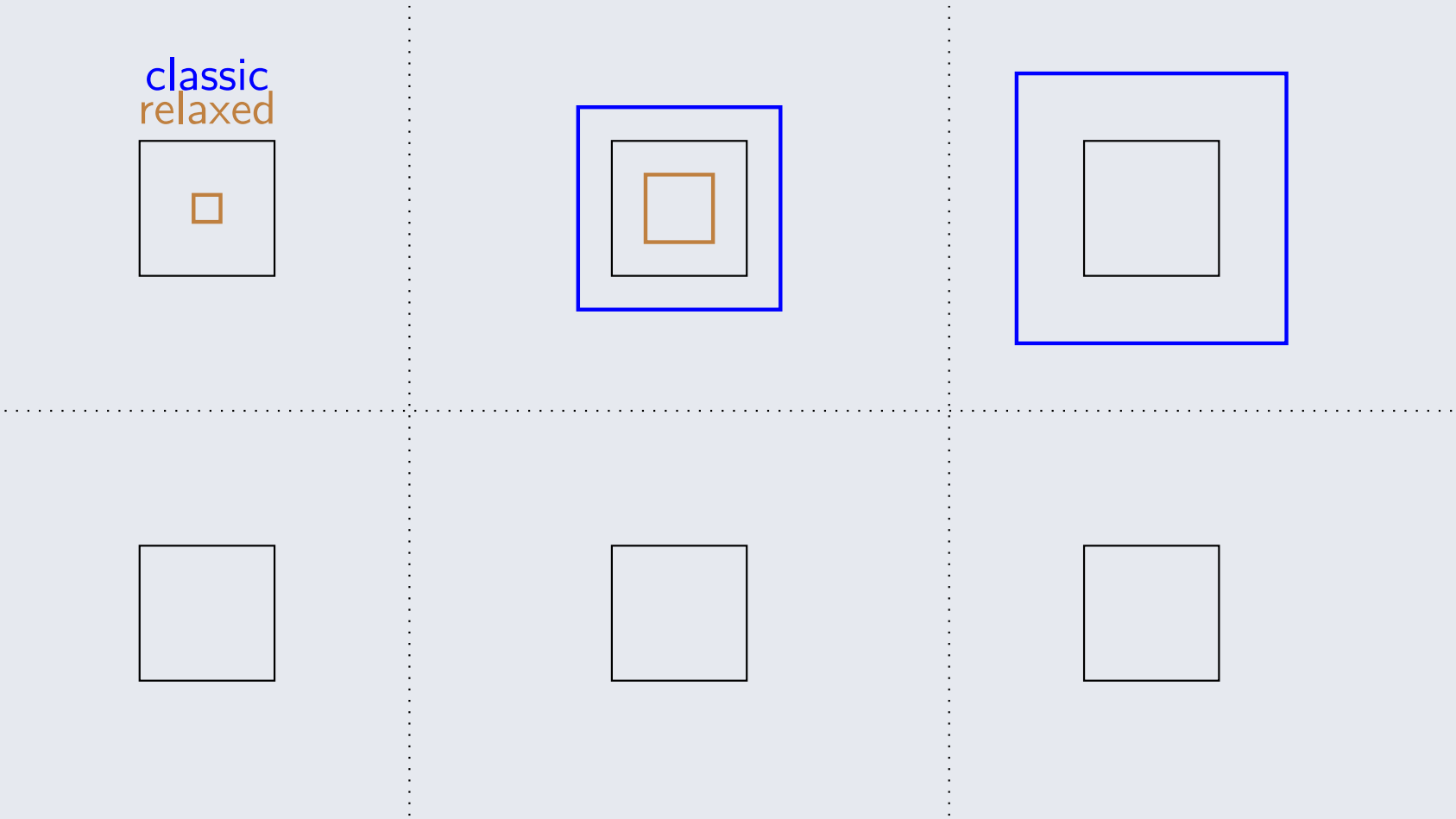

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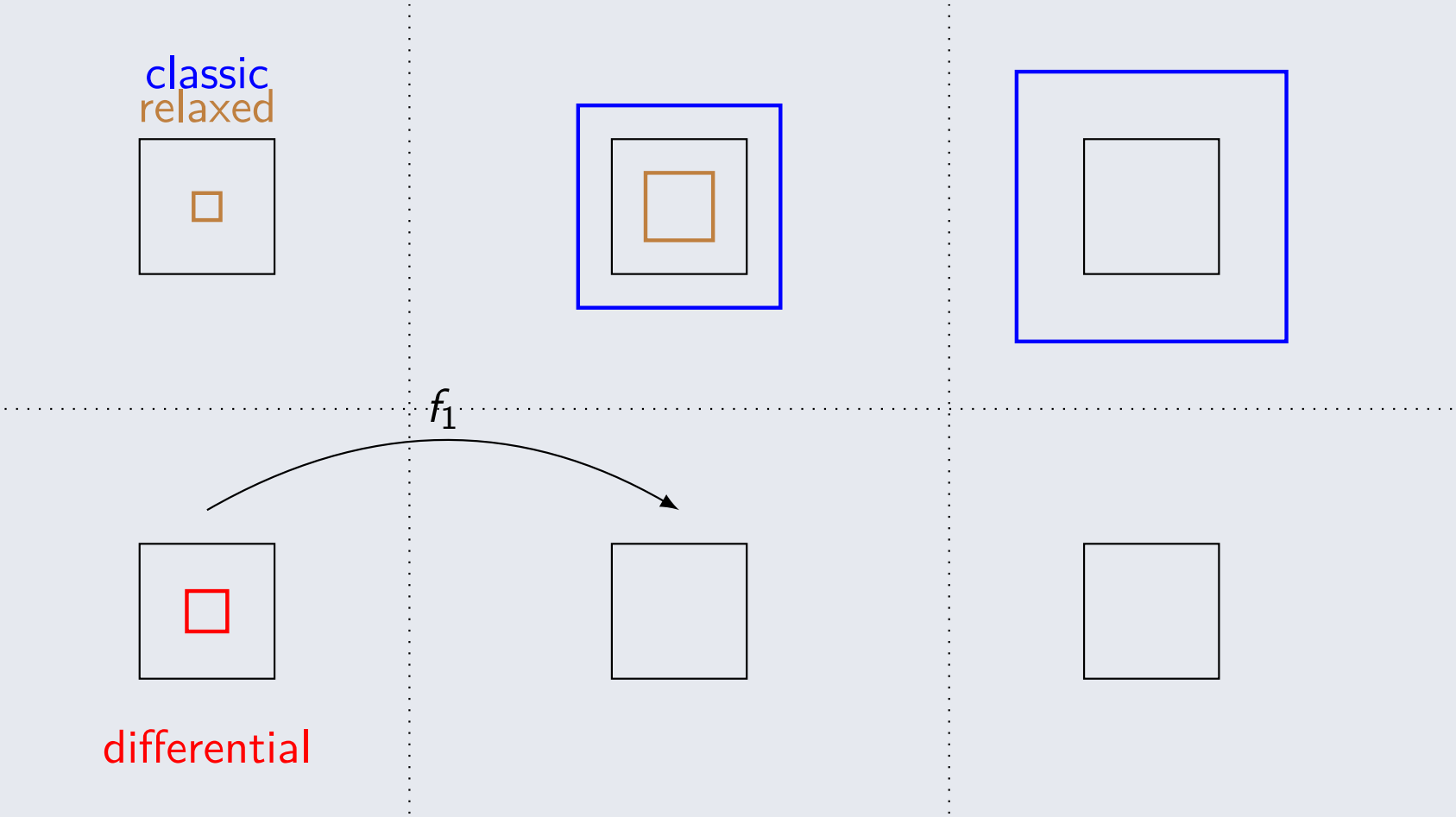

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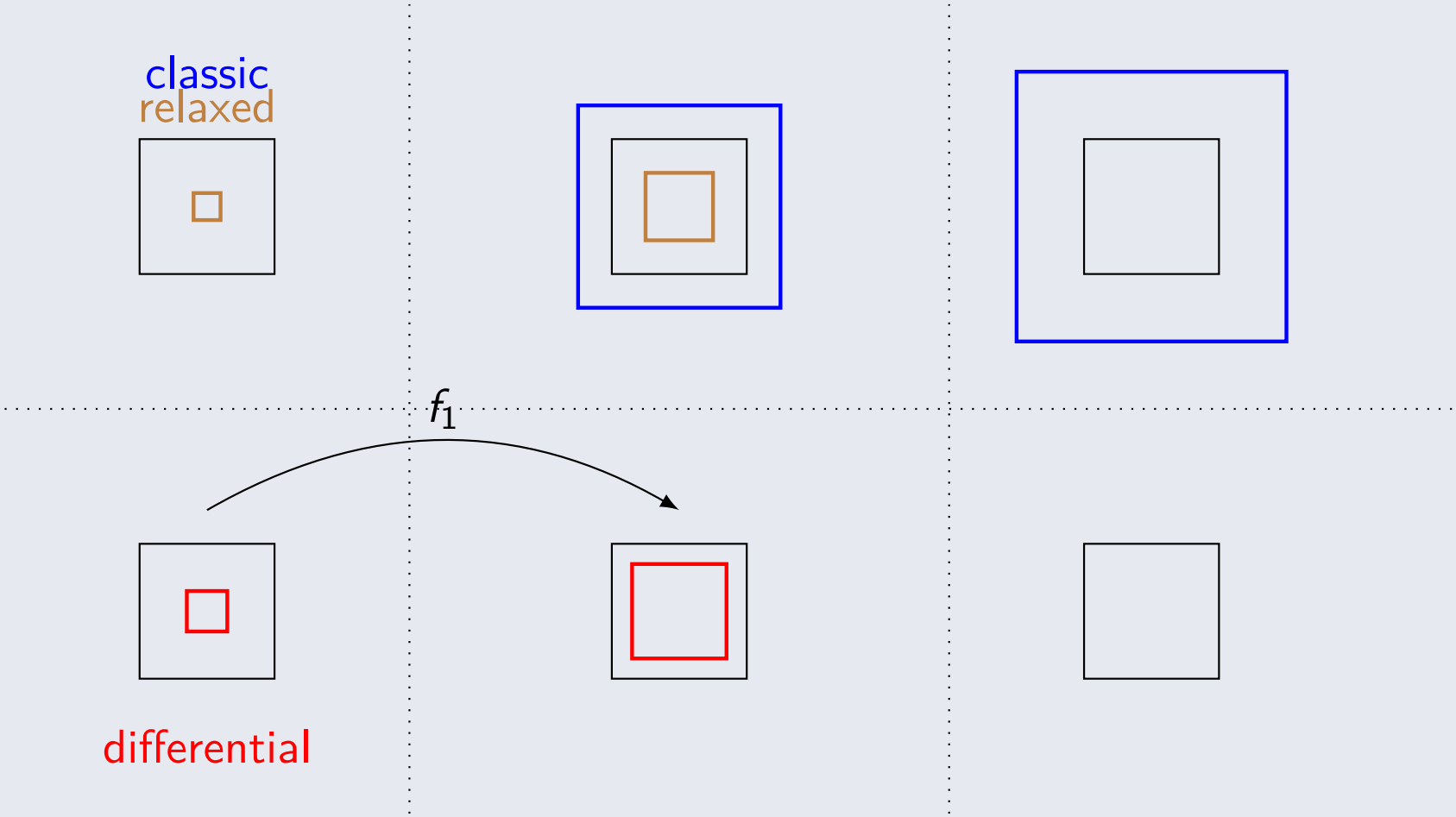

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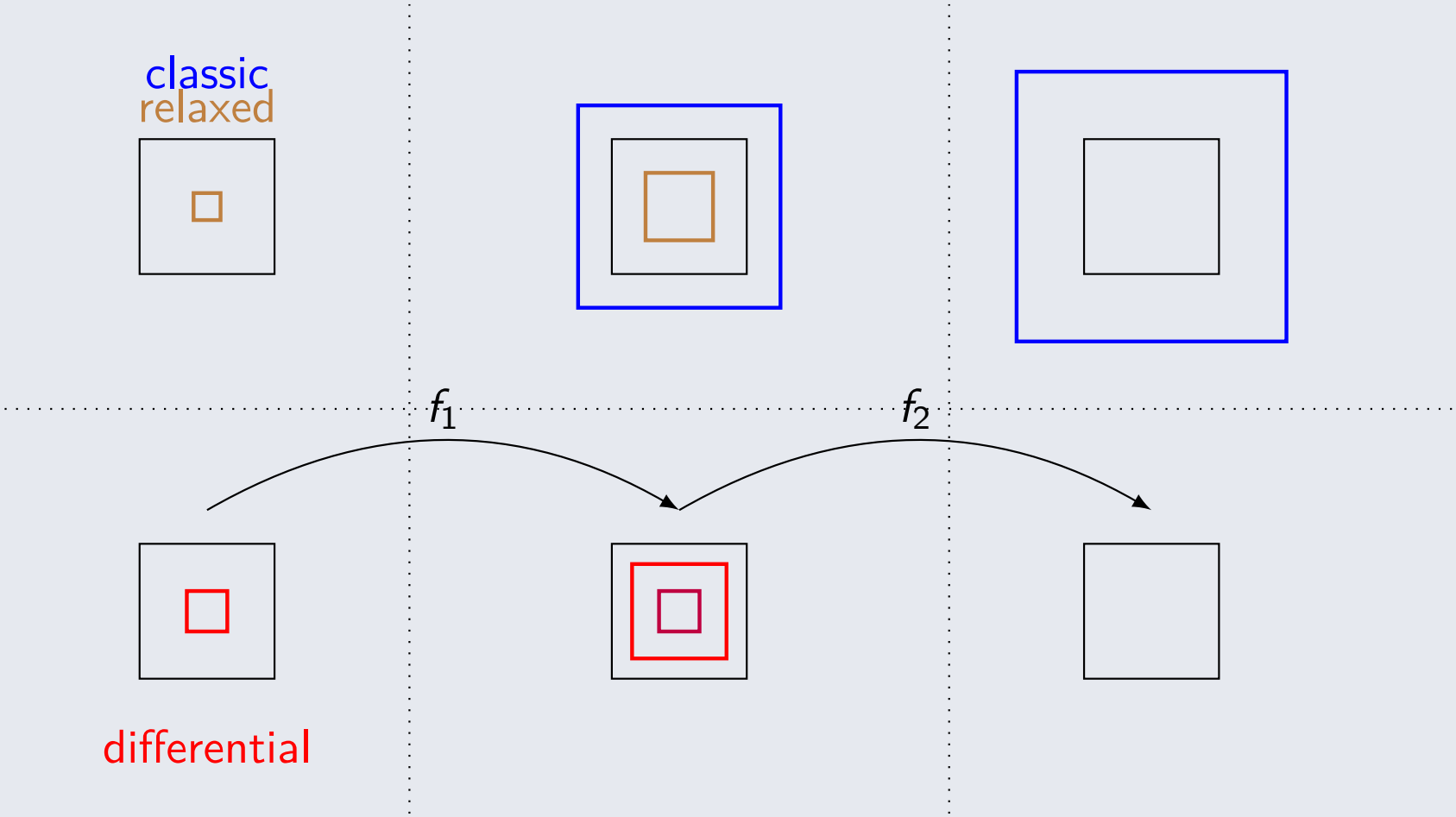

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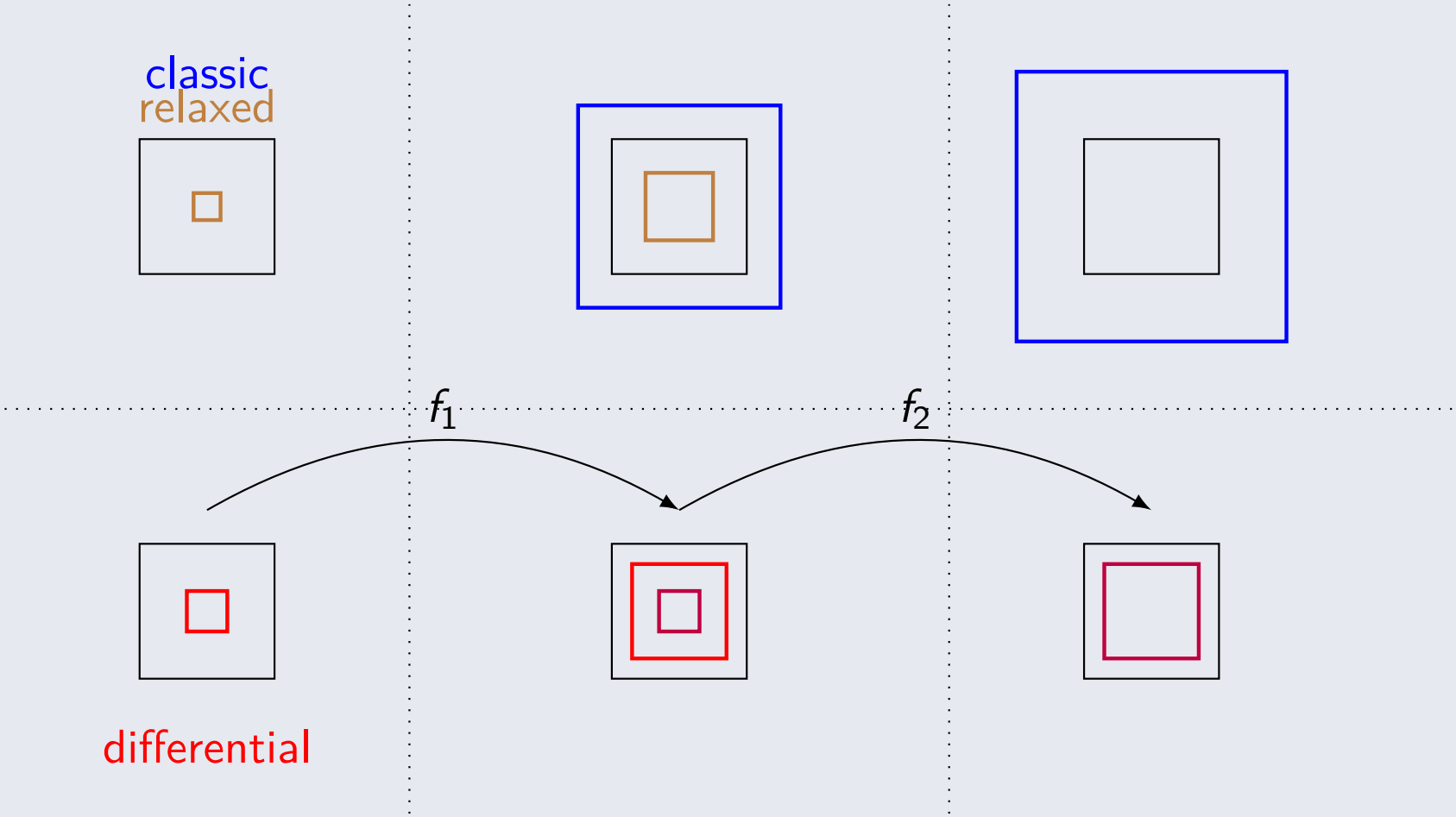

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Therefore, $\delta Q$ and $\delta R$ are determined by the division of $\delta A - Q\delta B$ by $B$.

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\begin{aligned}
M &= LU \\
M + \delta M &= (L + \delta L)(U + \delta U) \\
M + \delta M &= LU + \delta L \times U + L \times \delta U \\
\delta M &= \delta L \times U + L \times \delta U \\
L^{-1} \times \delta M \times U^{-1} &= L^{-1} \times \delta L + \delta U \times U^{-1}
\end{aligned}
$$

# About matrices

## Differential of the LU factorization

We would like to differentiate $M \mapsto (L, U)$.

$$\begin{aligned} M &= LU \\ M + \delta M &= (L + \delta L)(U + \delta U) \\ M + \delta M &= LU + \delta L \times U + L \times \delta U \\ \delta M &= \delta L \times U + L \times \delta U \\ L^{-1} \times \delta M \times U^{-1} &= L^{-1} \times \delta L + \delta U \times U^{-1} \end{aligned}$$

Therefore, $\begin{aligned} \delta L &= L \times \left(L^{-1} \times \delta M \times U^{-1}\right)_{\text{Low}} \\ \delta U &= \left(L^{-1} \times \delta M \times U^{-1}\right)_{\text{Up}} \times U \end{aligned}$

# To sum up

## On $p$-adic precision

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## On $p$-adic precision

- Step-by-step analysis : as a first step. Can show differentiability.
- Differential calculus : **intrinsic** and can handle both **gain** and **loss**.
- New framework : differentials and lattices.

## Future works

- Implement automatic computation of differentials, within the computation.

# References

## Over $p$-adic analysis

- Peter Schneider $p$-adic Lie groups. (Springer).

## Over $p$-adic precision

- Xavier Caruso Random matrix over a DVR and LU factorization, preprint.
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