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The Koszul complex $K_\bullet(\underline{a}; \mathbf{A})$ or its dual $K^\bullet(\underline{a}; \mathbf{A})$ leads to the definition of depth:

$$\begin{aligned} \text{Gr}(\underline{a}; \mathbf{A}) \geq k &\iff H^i(\underline{a}; \mathbf{A}) = 0, \forall i = 0, 1, \dots, k-1 \\ &\iff H_i(\underline{a}; \mathbf{A}) = 0, \forall i = n, n-1, \dots, n-k+1 \end{aligned}$$

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What makes a complex exact ?

Buchsbaum-Eisenbud criterion

We assume given a complex of free modules :

$$0 \rightarrow F_m \xrightarrow{u_m} F_{m-1} \rightarrow \cdots \rightarrow F_1 \xrightarrow{u_1} F_0$$

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with $F_k = \mathbf{A}^{\nu_k}$. We define the sequence (r_1, \dots, r_m) such that

$$r_m = \nu_m, \quad r_{m-1} = \nu_{m-1} - r_m, \quad \dots \quad r_1 = \nu_1 - r_2$$

If $\text{Gr}(\mathcal{D}_{r_k}(u_k)) \geq k$ for $k = m, \dots, 1$, then the complex is exact.
The converse is also true.

The three points in \mathbb{P}^2

Example : $\mathbf{S} = \mathbf{k}[X, Y, Z]$, $\mathbf{a} = \langle XY, XZ, YZ \rangle$

Let $M = (a_{ij}) \in \mathbb{M}_{n,n-1}(\mathbf{A})$ be a matrix. We define $\Delta_1, \dots, \Delta_n$ which are, up to sign, the minors of M of size $n-1$:

$$\begin{vmatrix} X_1 & a_{11} & \cdots & a_{1n-1} \\ X_2 & a_{21} & \cdots & a_{2n-1} \\ \vdots & \vdots & & \vdots \\ X_n & a_{n1} & \cdots & a_{nn-1} \end{vmatrix} = \Delta_1 X_1 + \cdots + \Delta_n X_n$$

Since $[\Delta_1 \ \cdots \ \Delta_n] M = [0 \ \cdots \ 0]$, we have the *complex* of free modules :

$$0 \longrightarrow \mathbf{A}^{n-1} \xrightarrow{M} \mathbf{A}^n \xrightarrow{[\Delta_1 \ \cdots \ \Delta_n]} \mathbf{A}$$

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When is this complex exact ?

Corollary of the Buchsbaum-Eisenbud criterion

If $\text{Gr}((\Delta_1, \dots, \Delta_n); \mathbf{A}) \geq 2$, then the complex is exact.

The quartic rational curve in \mathbb{P}^3

$$\mathbf{S} = \mathbf{k}[X_0, \dots, X_3],$$

$$\mathfrak{a} = \langle X_1^3 - X_0^2 X_2, X_0 X_2^2 - X_1^2 X_3, X_2^3 - X_1 X_3^2, X_1 X_2 - X_0 X_3 \rangle$$

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$$0 \longrightarrow \mathbf{S} \xrightarrow{d_3} \mathbf{S}^4 \xrightarrow{d_2} \mathbf{S}^4 \xrightarrow{d_1} \mathbf{S} \longrightarrow \mathbf{S}/\mathfrak{a}$$

$$d_3 = \begin{bmatrix} X_3 \\ -X_2 \\ X_1 \\ X_0 \end{bmatrix} \quad \text{and} \quad d_2 = \begin{bmatrix} X_2 & X_3 & 0 & 0 \\ X_0 & X_1 & X_2 & -X_3 \\ 0 & 0 & -X_0 & X_1 \\ X_1^2 & X_0 X_3 & -X_1 X_3 & X_2^2 \end{bmatrix}$$

Let \mathbf{k} be a field, $\mathbf{S} = \mathbf{k}[X_1, \dots, X_n]$ and \mathfrak{a} a homogeneous ideal.
We focus on the graded finitely presented \mathbf{k} -algebra

$$\mathbf{S}/\mathfrak{a} = \mathbf{k}[x_1, \dots, x_n]$$

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- $\text{Kdim}(\mathbf{S}/\mathfrak{a})$: the Krull dimension of \mathbf{S}/\mathfrak{a}

Theorem

Let \mathfrak{a} be a homogeneous ideal of $\mathbf{S} = \mathbf{k}[X_1, \dots, X_n]$ where \mathbf{k} is a field. Then

$$n = \text{Gr}(\underline{x}; \mathbf{S}/\mathfrak{a}) + \text{Pd}(\mathbf{S}/\mathfrak{a})$$

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Krull dimension and Noether normalisation

There exists an explicit way to compute the Krull dimension of \mathbf{S}/\mathfrak{a} : the Noether normalisation, i.e.

d homogeneous elements h_1, \dots, h_d of \mathbf{S}/\mathfrak{a} such that

- (i) h_1, \dots, h_d are algebraically independent over \mathbf{k}
- (ii) \mathbf{S}/\mathfrak{a} is integral over $\mathbf{k}[h_1, \dots, h_d]$ (equivalently, \mathbf{S}/\mathfrak{a} is a $\mathbf{k}[h_1, \dots, h_d]$ -module finitely generated)

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Remark :

- (i) $\Leftrightarrow \mathbf{k}[H_1, \dots, H_d] \cap \mathfrak{a} = 0$ where $H_i \in \mathbf{S}$ such that $\overline{H_i} = h_i$
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We say that (h_1, \dots, h_d) is a homogenous system of parameters (h.s.o.p.) of \mathbf{S}/\mathfrak{a} .

The three points in \mathbb{P}^2

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What is the Krull dimension expected for \mathbf{S}/\mathfrak{a} ?

\mathfrak{a} is the ideal of three points in \mathbb{P}^2 , so

$$\text{Kdim}(\mathbf{S}/\mathfrak{a}) = \begin{array}{c} 0 \\ \uparrow \\ \text{points} \end{array} + \begin{array}{c} 1 \\ \uparrow \\ \text{projective} \\ \text{geometry} \end{array}$$

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Remark referring to Claude's presentation : \mathfrak{a} is a monomial ideal. Its simplicial complex of Stanley-Reisner consists of three isolated points.

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We have

$$\text{Kdim}(\mathbf{S}/\mathfrak{a}) = 1$$

We find a homogeneous element $h \in \mathbf{S}/\mathfrak{a}$ such that \mathbf{S}/\mathfrak{a} is integral over $\mathbf{k}[h]$.

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What about $\text{Gr}((x, y, z); \mathbf{S}/\mathfrak{a}) \leq \text{Kdim}(\mathbf{S}/\mathfrak{a})$?

The rational quartic curve in \mathbb{P}^3

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We have

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The rational quartic curve in \mathbb{P}^3

$\mathbf{S}/\mathfrak{a} = \mathbf{k}[x_0, x_1, x_2, x_3]$ with

$$x_1^3 = x_0^2 x_2, \quad x_0 x_2^2 = x_1^2 x_3, \quad x_2^3 = x_1 x_3^2, \quad x_1 x_2 = x_0 x_3$$

We propose $h_1 = x_0$ and $h_2 = x_3$ as h.s.o.p of \mathbf{S}/\mathfrak{a} .

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Here, we have $1 = \text{Gr}(\underline{x}; \mathbf{S}/\mathfrak{a}) < \text{Kdim}(\mathbf{S}/\mathfrak{a}) = 2$.

A last example

$$\mathbf{S} = \mathbf{k}[x_1, \dots, x_6]$$

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Resolution of \mathbf{S}/\mathfrak{a} ?

If $\text{characteristic}(\mathbf{k}) \neq 2$, for example $\mathbf{k} = \mathbb{Q}$, then

$$0 \longrightarrow \mathbf{S}^6 \longrightarrow \mathbf{S}^{15} \longrightarrow \mathbf{S}^{10} \longrightarrow \mathbf{S} \twoheadrightarrow \mathbf{S}/\mathfrak{a}$$

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If $\text{characteristic}(\mathbf{k}) = 2$, for example $\mathbf{k} = \mathbb{F}_2$, then

$$0 \longrightarrow \mathbf{S} \longrightarrow \mathbf{S}^7 \longrightarrow \mathbf{S}^{15} \longrightarrow \mathbf{S}^{10} \longrightarrow \mathbf{S} \longrightarrow \mathbf{S}/\mathfrak{a}$$

$$\begin{bmatrix}
 -x_4 & -x_5 & \cdot & \cdot & \cdot & -x_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 x_1 & \cdot & \cdot & \cdot & -x_5 & \cdot & \cdot & \cdot & -x_6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & x_3 & -x_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -x_6 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & x_2 & -x_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -x_6 & \cdot & \cdot \\
 \cdot & \cdot & \cdot & x_1 & x_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -x_6 \\
 \cdot & \cdot & \cdot & \cdot & \cdot & x_2 & \cdot & -x_4 & \cdot & \cdot & -x_5 & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -x_2 & x_3 & \cdot & \cdot & \cdot & \cdot & x_5 & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x_1 & \cdot & x_3 & \cdot & \cdot & \cdot & \cdot & -x_5 & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x_1 & \cdot & -x_3 & \cdot & x_4 & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x_1 & x_2 & \cdot & \cdot & x_4
 \end{bmatrix}$$

$$d_3 = \begin{bmatrix} x_5 & x_6 & \cdot & \cdot & \cdot & \cdot \\ -x_4 & \cdot & x_6 & \cdot & \cdot & \cdot \\ -x_3 & \cdot & \cdot & x_6 & \cdot & \cdot \\ -x_2 & \cdot & \cdot & \cdot & x_6 & \cdot \\ x_1 & \cdot & \cdot & \cdot & \cdot & x_6 \\ \cdot & -x_4 & -x_5 & \cdot & \cdot & \cdot \\ \cdot & -x_3 & \cdot & x_5 & \cdot & \cdot \\ \cdot & -x_2 & \cdot & \cdot & x_5 & \cdot \\ \cdot & x_1 & \cdot & \cdot & \cdot & -x_5 \\ \cdot & \cdot & x_3 & -x_4 & \cdot & \cdot \\ \cdot & \cdot & -x_2 & \cdot & -x_4 & \cdot \\ \cdot & \cdot & x_1 & \cdot & \cdot & -x_4 \\ \cdot & \cdot & \cdot & x_2 & -x_3 & \cdot \\ \cdot & \cdot & \cdot & x_1 & \cdot & -x_3 \\ \cdot & \cdot & \cdot & \cdot & x_1 & x_2 \end{bmatrix}$$

We have $d_2(v) = 0$ and $2v = d_3(w)$ with

$$v = \begin{bmatrix} \cdot \\ x_4x_6 \\ x_3x_6 \\ x_2x_6 \\ \cdot \\ -x_4x_5 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ -x_2x_4 \\ \cdot \\ \cdot \\ \cdot \\ x_1x_2 \end{bmatrix} \quad w = \begin{bmatrix} -x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}$$

A last example

$$\mathbf{S} = \mathbf{k}[x_1, \dots, x_6]$$

$$\mathfrak{a} = \langle x_1 x_2 x_3, x_2 x_3 x_4, x_1 x_2 x_5, x_1 x_4 x_5, x_3 x_4 x_5, \\ x_1 x_3 x_6, x_1 x_4 x_6, x_2 x_4 x_6, x_2 x_5 x_6, x_3 x_5 x_6 \rangle$$

Krull dimension of \mathbf{S}/\mathfrak{a} ?

A last example

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Krull dimension of \mathbf{S}/\mathfrak{a} ?

\mathfrak{a} is a monomial ideal : its Stanley-Reisner simplicial complex is of dimension 2, so $\text{Kdim}(\mathbf{S}/\mathfrak{a}) = 2 + 1 = 3$.

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Krull dimension of \mathbf{S}/\mathfrak{a} ?

\mathfrak{a} is a monomial ideal : its Stanley-Reisner simplicial complex is of dimension 2, so $\text{Kdim}(\mathbf{S}/\mathfrak{a}) = 2 + 1 = 3$.

We have an h.s.o.p.

$$h_1 = x_1 + x_3 + x_4, \quad h_2 = x_2 + x_3 + x_6, \quad h_3 = x_2 + x_4 + x_5$$