

Depth, Dimension and Resolutions in Commutative Algebra

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- This morning: the Koszul complex, regular sequence, depth
- Tomorrow: the Buchsbaum & Eisenbud criterion and the equality of Auslander & Buchsbaum **through examples**.
- Wednesday: some elementary results about the homology of a bicomplex

I will begin with a little example. . .

Let us consider the ideal $\mathfrak{a} = \langle X_1, X_2, X_3 \rangle$ of $\mathbf{A} = \mathbf{k}[X_1, X_2, X_3]$.
What is "the" resolution of \mathbf{A}/\mathfrak{a} as \mathbf{A} -module? (the question is
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$$\cdots \longrightarrow F_m \xrightarrow{d_m} F_{m-1} \xrightarrow{d_{m-1}} \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \twoheadrightarrow \mathbf{A}/\mathfrak{a}$$

with \mathbf{A} -modules F_i as simple as possible and s.t. $\text{Im } d_i = \text{Ker } d_{i-1}$.

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We say that F_\bullet is a **resolution** of the \mathbf{A} -module \mathbf{A}/\mathfrak{a} of length m .

The Koszul complex of a sequence $\underline{a} = (a_1, \dots, a_n)$

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If $\text{Im } \partial_{d+1} = \text{Ker } \partial_d$, we say that the complex is **exact** in degree d .

Here is the Koszul complex for a sequence of length 4 (where, for every degree d , the basis $(e_I)_{\#I=d}$ is ordered lexicographically)

$$\mathbf{A} \xrightarrow{\begin{bmatrix} -a_4 \\ a_3 \\ -a_2 \\ a_1 \end{bmatrix}} \mathbf{A}^4 \xrightarrow{\begin{bmatrix} a_3 & a_4 & \cdot & \cdot \\ -a_2 & \cdot & a_4 & \cdot \\ \cdot & -a_2 & -a_3 & \cdot \\ a_1 & \cdot & \cdot & a_4 \\ \cdot & a_1 & \cdot & -a_3 \\ \cdot & \cdot & a_1 & a_2 \end{bmatrix}} \mathbf{A}^6 \xrightarrow{\begin{bmatrix} -a_2 & -a_3 & -a_4 & \cdot & \cdot & \cdot \\ a_1 & \cdot & \cdot & -a_3 & -a_4 & \cdot \\ \cdot & a_1 & \cdot & a_2 & \cdot & -a_4 \\ \cdot & \cdot & a_1 & \cdot & a_2 & a_3 \end{bmatrix}} \mathbf{A}^4 \xrightarrow{[a_1 \ a_2 \ a_3 \ a_4]} \mathbf{A}$$

If we want to keep in mind one key sentence: " $K_\bullet(\underline{a}; \mathbf{A})$ is the complex of the trivial syzygies of \underline{a} ".

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In a more structural way, the Koszul complex is defined by the exterior algebra

$$0 \longrightarrow \bigwedge^n(\mathbf{A}^n) \xrightarrow{\partial_n} \dots \longrightarrow \bigwedge^1(\mathbf{A}^n) \xrightarrow{\partial_1} \bigwedge^0(\mathbf{A}^n) \longrightarrow 0$$

and the differential ∂ is defined by

$$\partial(x_1 \wedge \dots \wedge x_d) = \sum_{i=1}^d (-1)^{i-1} \mu(x_i) x_1 \wedge \dots \wedge \widehat{x}_i \wedge \dots \wedge x_d$$

where $\mu : \mathbf{A}^n \xrightarrow{[a_1 \ \dots \ a_n]} \mathbf{A}$

Regular sequence

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For $\underline{a} = (X_1, X_2, X_3)$ and $\mathbf{A} = \mathbf{k}[X_1, X_2, X_3]$, we "saw" that $K_\bullet(\underline{a}; \mathbf{A})$ is exact (except in degree 0 of course, unless $1 = 0$ in \mathbf{k}).

More generally, the Koszul complex of a regular sequence \underline{a} is exact in degrees > 0 . Rmk : exactness in degree 0 means $1 \in \langle \underline{a} \rangle$

Definition

We say that \underline{a} is a regular sequence when

- a_1 is a regular element (non zero divisor in \mathbf{A})
- a_2 is a regular element modulo $\langle a_1 \rangle$ (non zero divisor in $\mathbf{A}/\langle a_1 \rangle$)
- \vdots
- a_i is a regular element modulo $\langle a_1, \dots, a_{i-1} \rangle$

Example : let's take $n = 3 \dots$

Warning. We have :

$$\underline{a} \text{ regular} \Rightarrow K_{\bullet}(\underline{a}; \mathbf{A}) \text{ exact in degrees } > 0$$

but the converse is not true.

In general, a regular sequence needs not remain regular under permutation. Let us consider the famous example : $\mathbf{A} = \mathbf{k}[X, Y, Z]$

$$a_1 = (Y - 1)X, \quad a_2 = Y, \quad a_3 = (Y - 1)Z,$$

We can prove that $\underline{a} = (a_1, a_2, a_3)$ is regular but not $\underline{b} = (a_1, a_3, a_2)$ (we have $Xa_3 \in \langle a_1 \rangle$ but $X \notin \langle a_1 \rangle$).

Nevertheless, we have $K_{\bullet}(\underline{a}; \mathbf{A}) \simeq K_{\bullet}(\underline{b}; \mathbf{A})$.

Proposition

$$\underline{a} = (a_1, \dots, a_n)$$

If $\langle \underline{a} \rangle$ contains a regular sequence of length k , then $K_\bullet(\underline{a}; \mathbf{A})$ is exact in degrees $n, n-1, \dots, n-(k-1)$

$$0 \longrightarrow K_n \xrightarrow{\partial_n} K_{n-1} \longrightarrow \dots \longrightarrow K_{n-k+1} \longrightarrow K_{n-k} \longrightarrow \dots$$

In other words $H_i(\underline{a}; \mathbf{A}) = 0$ for $i = n, n-1, \dots, n-k+1$.
where $H_i(\underline{a}; \mathbf{A}) := \text{Ker } \partial_i / \text{Im } \partial_{i+1}$

It is often more convenient to use the dual of $K_\bullet(\underline{a}; \mathbf{A})$ to measure the regularity.

$$n, n-1, \dots, n-k+1 \quad \rightsquigarrow \quad 0, 1, \dots, k-1$$

The cochain Koszul complex $K^\bullet(\underline{a}; \mathbf{A})$

We define the cochain Koszul complex $K^\bullet(\underline{a}; \mathbf{A})$:

$$0 \longrightarrow \bigwedge^0(\mathbf{A}^n) \xrightarrow{\delta} \bigwedge^1(\mathbf{A}^n) \xrightarrow{\delta} \dots \longrightarrow \bigwedge^n(\mathbf{A}^n) \longrightarrow 0$$

where $\delta = \underline{a} \wedge \bullet$. In particular, we have

$$\delta(e_I) = \sum_{i \notin I} (-1)^{\varepsilon_i(I)} a_i e_{I \vee i}$$

where $\varepsilon_i(I)$ is the number of elements of I less than i .

Self-duality of the Koszul complex

Theorem

We have $K_{\bullet}(\underline{a}; \mathbf{A}) \simeq K^{\bullet}(\underline{a}; \mathbf{A})$. In other words, the Koszul complex is isomorphic to its dual.

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We have $K_*(\underline{a}; \mathbf{A}) \simeq K^*(\underline{a}; \mathbf{A})$. In other words, the Koszul complex is isomorphic to its dual.

To convince you (?) – in fact, it is not very clear (there are signs, and it is quite laborious if you don't know, a priori, how find an isomorphism) –, here is the Koszul complex $K_*(\underline{a}; \mathbf{A})$ and below its dual $K^*(\underline{a}; \mathbf{A})$:

$$\begin{array}{ccccccc}
 \Lambda^4 & \xrightarrow{\begin{bmatrix} -a_4 \\ a_3 \\ -a_2 \\ a_1 \end{bmatrix}} & \Lambda^3 & \xrightarrow{\begin{bmatrix} a_3 & a_4 & \cdot & \cdot \\ -a_2 & \cdot & a_4 & \cdot \\ \cdot & -a_2 & -a_3 & \cdot \\ a_1 & \cdot & \cdot & a_4 \\ \cdot & a_1 & \cdot & -a_3 \\ \cdot & \cdot & a_1 & a_2 \end{bmatrix}} & \Lambda^2 & \xrightarrow{\begin{bmatrix} -a_2 & -a_3 & -a_4 & \cdot & \cdot & \cdot \\ a_1 & \cdot & \cdot & -a_3 & -a_4 & \cdot \\ \cdot & a_1 & \cdot & a_2 & \cdot & -a_4 \\ \cdot & \cdot & a_1 & \cdot & a_2 & a_3 \end{bmatrix}} & \Lambda^1 & \xrightarrow{[a_1 \ a_2 \ a_3 \ a_4]} & \Lambda^0 \\
 \\
 \Lambda^0 & \xrightarrow{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}} & \Lambda^1 & \xrightarrow{\begin{bmatrix} -a_2 & a_1 & \cdot & \cdot \\ -a_3 & \cdot & a_1 & \cdot \\ -a_4 & \cdot & \cdot & a_1 \\ \cdot & -a_3 & a_2 & \cdot \\ \cdot & -a_4 & a_2 & \cdot \\ \cdot & \cdot & -a_4 & a_3 \end{bmatrix}} & \Lambda^2 & \xrightarrow{\begin{bmatrix} a_3 & -a_2 & \cdot & a_1 & \cdot & \cdot \\ a_4 & \cdot & -a_2 & \cdot & a_1 & \cdot \\ \cdot & a_4 & -a_3 & \cdot & \cdot & a_1 \\ \cdot & \cdot & \cdot & a_4 & -a_3 & a_2 \end{bmatrix}} & \Lambda^3 & \xrightarrow{[-a_4 \ a_3 \ -a_2 \ a_1]} & \Lambda^4
 \end{array}$$

Therefore $H_i(\underline{a}; \mathbf{A}) \simeq H^{n-i}(\underline{a}; \mathbf{A})$.

$H^0(\underline{a}; \mathbf{A})$

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It is the kernel of :

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Thus

$$H^0(\underline{a}; \mathbf{A}) = \text{Ann}(\underline{a}) = \{x \in \mathbf{A} \mid a_i x = 0, \forall i\}$$

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$$\boxed{H^0(\underline{a}; \mathbf{A}) = 0 \Leftrightarrow \text{Ann}(\underline{a}) = 0}$$

What means the exactness of $K^\bullet(\underline{a}; \mathbf{A})$ in cohomological degree 1?

$$\Lambda^0(\mathbf{A}^n) \xrightarrow{\delta_1} \Lambda^1(\mathbf{A}^n) \xrightarrow{\delta_2} \Lambda^2(\mathbf{A}^n) \longrightarrow \dots$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \wedge \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

We have $H^1(\underline{a}; \mathbf{A}) = 0$ iff

$\forall (x_1, \dots, x_n) \in \mathbf{A}^n$ such that $\forall i, j, a_i x_j = a_j x_i$, then
 $\exists y \in \mathbf{A}$ such that $x_i = a_i y, \forall i$.

Definition of depth

We define the depth of \underline{a} by the vanishing of the Koszul homology (or equivalently cohomology) of \underline{a} . We write:

$$\begin{aligned}\operatorname{Gr}(\underline{a}; \mathbf{A}) \geq k &\iff H^i(\underline{a}; \mathbf{A}) = 0, \forall i = 0, 1, \dots, k-1 \\ &\iff H_i(\underline{a}; \mathbf{A}) = 0, \forall i = n, n-1, \dots, n-k+1\end{aligned}$$

The depth is closely related to the notion of regular sequence

(if $\langle \underline{a} \rangle$ contains a regular sequence of length k , then $\operatorname{Gr}(\underline{a}; \mathbf{A}) \geq k$)

but it is not the same notion:

$H^0(\underline{a}; \mathbf{A}) = \operatorname{Ann}(\underline{a}) = 0 \not\Rightarrow$ existence of a regular element in $\langle \underline{a} \rangle$

Example

$\mathbf{A} = \mathbf{k}[X, Y, Z]$ and $\underline{a} = (XY, XZ, YZ)$.

$\text{Gr}(\underline{a}; \mathbf{A}) \cong ?$

We can define the Koszul complex of a sequence \underline{a} with respect to a \mathbf{A} -module M :

$$K_{\bullet}(\underline{a}; M) := K_{\bullet}(\underline{a}; \mathbf{A}) \otimes_{\mathbf{A}} M$$

Idem for $K^{\bullet}(\underline{a}; M)$.

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Tomorrow, $\mathbf{A} = \mathbf{k}[\underline{X}] = \mathbf{k}[X_1, \dots, X_n]$ and $M = \mathbf{A}/\mathfrak{a}$ with \mathfrak{a} an *homogeneous* ideal of \mathbf{A} . And we will focus on $\text{Gr}(\underline{X}; \mathbf{k}[\underline{X}]/\mathfrak{a})$.

Example : $\mathbf{A} = \mathbf{k}[X, Y, Z]$, $\mathfrak{a} = \langle XY, XZ, YZ \rangle$.

$$\text{Gr}((X, Y, Z); \mathbf{A}/\mathfrak{a}) \cong ?$$