

A theoretical study of steerable homogeneous operators, and applications

Research Internship Report – Master MVA

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Internship with the BIG team at EPFL, Lausanne (Switzerland)
April – August 2016

* If needed, see on-line at <http://lbo.k.vu/epfl2016> for a PDF version of this report, and additional resources (slides, code, figures, complete bibliography etc), open-sourced under the MIT License. Note that this document is my internship report, but also my Master thesis.

Abstract

This report presents the research I did between April and August 2016, in theoretical function analysis and operator theory, at the BIG team at EPFL.

We focus mainly on two aspects: convolution operators in any dimension; and steerable homogeneous convolutions in two dimensions (for images), and their possible applications. A lot of our results are valid regardless of the dimensions, so we stay in a general setting as long as we can, and then we restrict to operators on 2D images for the study of steerable operators.

We start by recalling the common notations used in signal processing and functional analysis, and then by recalling the main properties of a fundamental tool for this domain, the Fourier transform \mathcal{F} . We expose the main goals of our research, in order to motivate this theoretical study of convolution operators also from a practical point of view.

We chose to follow a very didactic approach, and so we almost redefine “from scratch” the theory of functional operators, along with proofs of its most important results. Our operators can have structural and geometric properties, namely linearity or continuity, and translation-, scaling-, rotation-invariance, unity – all these properties being already well-known – or steerability. We study extensively the links between all these properties, and we present many characterizations. To the best of our knowledge, this document is the first attempt to summarize all these results, and some of our latest characterizations seemed to be new results.

After a very broad section on operators, we focus on steerable convolutions G , mainly in $2D$, as they appear to be the natural framework for image analysis tasks. Our main results consist in characterizations of steerable convolutions, first written as a sum of modulated and iterated real Riesz transforms. Then adding the γ -scale-invariance gives a nicer form, as a composition of a fractional Laplacian $(-\Delta)^{\gamma/2}$, some directional derivatives D_{α_i} and a “nice” part G_{nice} ; and another form as a composition of elementary blocks that all have the same form $G_{\lambda,\alpha}$. This last form is very appealing for implementation, as it is enough to program the elementary block, and to compose it to obtain every 2D steerable γ -SI convolution, and it has a strong theoretical interpretation: a G N -SI and steerable of order n_G gets decomposed as a product of N elementary blocks, all 1-SI steerable of order 1 or 2. And a simple universality theorem on $G_{\lambda,\alpha}$ reinforces this result: the elementary blocks are exactly the 1-SI steerable convolutions of order 1 or 2.

We conclude by explaining how to implement our operators and their inverse, and by presenting the results of some experiments on 2D stochastic processes, in order to illustrate the effects of our elementary blocks as well as more complicated operators. We highlight some properties on the examples, like the trade-of between the directionality of D_{α_i} and the isotropy of $(-\Delta)^{\gamma/2}$. Our operators could also be used to develop new splines (for new sampling schemes), and new Green’s functions (for new denoising and data recovery algorithms), but we did not have the time to fully study these aspects.

We also study unitary operators extensively, but it is only included in an appendix.

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Internship advisors: Julien Fageot and Michael Unser

Location: LIB/BIG team, EPFL (Lausanne)

Master program: “Mathématiques, Vision, Apprentissage” (MVA)
in 2015–2016, at ENS de Cachan (France).

C Acknowledgments

I would like to first thank warmly Prof.Dr. Michael Unser, director of the BIG team at EPFL, for trusting me and accepting me in his team for this 5-month internship, and both him and Julien Fageot for their supervision, their advices and their constant help. Julien was very present and helpful, on every aspect of our research. Many thanks also to Nadia Macor for her kind and efficient support for all administrative aspects, and for her always cheerful attitude.

In France, thanks to Prof. Gabriel Peyré for accepting to be my internship advisor for the master MVA, reviewing my report and attending to the oral presentation; and at ENS de Cachan thanks to Delphine Laverne and Prof. Alain Trouvé for their administrative and financial support. Thanks also to Prof. Nicolas Vayatis for his advices while I was trying to chose my internship, and thanks to him and all the other professors I had in 2015–2016 during the master MVA.

At EPFL, thanks to my friend and desktop-mate⁸⁴ Thibault Groueix; and thanks also to all the members of the BIG team, especially to Harshit Gupta, Dr. Ferréol Soulez, Dr. Denis Fortun, Anaïs Badoual, Virginie Uhlmann, Silvia Colabrese, Laurène Donati and Dr. Daniel Sage for many fruitful discussions. Thanks to my roommates for their useful L^AT_EX-related suggestions, and many good moments we shared together.

My internship was partly funded by the mathematical department of ENS de Cachan in France, and by the Swiss European Mobility Program and the BIG team at EPFL in Switzerland – many thanks to them as well. Working at EPFL was a wonderful experience, made possible thanks to the Swiss European Mobility Program, and the help of Aurore Patey from the international relations office at ENS de Cachan.

Quick personal feeling about the internship

TL;DR: I enjoyed living in Lausanne and in Switzerland, I liked working at EPFL and with the BIGgers, and I loved working on this report during the last weeks of my internship.

I enjoyed very much working on this project, especially because it was a unique chance of working on such theoretical considerations of functional analysis and signal processing. It was exciting to have the time to work on the report and make it as complete and as detailed as I wanted. I hope it was not too long or too hard to follow!

We lacked the time to fully implement everything and to experiment in depth with stochastic sparse processes, and real images, but this is an interesting direction of research. Virginie, Julien and I will most surely keep working on this aspect, and hopefully we will soon try to publish our results.

Working in the BIG team at EPFL was very nice and intellectually stimulating, and I enjoyed staying in Lausanne during the spring and summer 2016. EPFL and its campus are quiet and nice to live in, and every aspect of my stay was made easy thanks to their great infrastructures.

This work is dedicated to the memory of the late Nicolas Pajor. ☹.

⁸⁴ For his daly help on basic maths, the regular experiments on Sublime Text, and command-line- or Ubuntu-related exploratory hacks that we used to enjoy together.

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The end. □