

Functions of several real variables (MA102, Part-I) ¹

1. Find the limits of following, if exist.

$$(a) \left(\sin\left(\frac{1}{n}\right), e^{-\frac{1}{n^2}}, \sin\left(\frac{\pi}{2} - \frac{1}{n}\right) \right) \quad (b) \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{(x+y+z)^2}{x^2+y^2+z^2} \quad (c) \lim_{(x,y) \rightarrow (0,0)} \frac{xy \cos(y)}{4x^2+y^2}$$

$$(d) \lim_{(x,y) \rightarrow (0,0)} \frac{3x^2|y|}{x^2+y^2} \quad (e) \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{1 - \cos(x+y+z)}{(x+y+z)^2} \quad (f) \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

2. Check the continuity of the following functions. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

(a) $f(x, y) = \frac{\sin(xy)}{xy}$, for $xy \neq 0$ and $f(x, y) = 1$ for $xy = 0$.

(b) $f(x, y) = \frac{x^3y}{2x^4+y^2}$, for $(x, y) \neq (0, 0)$ and $f(x, y) = 0$ for $(x, y) = (0, 0)$.

3. Let $f(x, y) = e^{\frac{-1}{|x-y|}}$, when $x \neq y$. How must f be defined for $x = y$ so that f is continuous on \mathbb{R}^2 ?

4. Let $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = |x| + |y|$ and $f(x, y) = |xy|$ for $(x, y) \in \mathbb{R}^2$. Show that

(a) $f_x(0, 0)$ and $f_y(0, 0)$ do not exist whereas $g_x(0, 0)$ and $g_y(0, 0)$ exist.

(b) for $x_0 \neq 0, g_y(x_0, 0)$ does not exist and for $y_0 \neq 0, g_x(0, y_0)$ does not exist.

5. Consider the function $f(x, y) = \frac{3x^2y-y^3}{x^2+y^2}$, for $(x, y) \neq (0, 0)$ and $f(x, y) = 0$ for $(x, y) = (0, 0)$.

(a) Verify whether f is continuous at $(0, 0)$ (b) Evaluate $f_y(x, 0)$ for $x \neq 0$ and (c) Verify whether f_y is continuous at $(0, 0)$.

6. In each of the following cases discuss the differentiability of f at $(0, 0)$ where $f(x, y)$, for $(x, y) \in \mathbb{R}^2$ is

(a) $|x| + |y|$ (b) $\|(x, y)\|$ (c) $f(x, y) = 0$, for $(x, y) \neq 0$ & $f(x, y) = 1$, for $(x, y) = 0$ (d) $\sqrt{|xy|}$

(e) $\frac{x^2y}{\sqrt{x^2+y^2}}$, $(x, y) \neq 0$ & $f(0, 0) = 0$ (f) $\frac{xy}{x^2+y^2}$, $(x, y) \neq 0$ & $f(0, 0) = 0$ (g) $x^2 + \sin(y) + y^2e^x$

7. Let $f(x, y) = |xy|$ for all $(x, y) \in \mathbb{R}^2$. Show that

(a) f is differentiable at $(0, 0)$ (b) $f_x(0, y_0)$ does not exist if $y_0 \neq 0$.

8. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) = (x - y)^2 \sin\left(\frac{1}{x-y}\right)$ if $x \neq y$ and $f(x, x) = 0$. Show that

(a) f_x and f_y exists at all points of \mathbb{R}^2 (b) f is differentiable at $(0, 0)$ (c) f_x and f_y are not continuous on the line $y = x$.

9. (Sufficient conditions for the differentiability) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $X_0 \in \mathbb{R}^2$ be such that all its partial derivatives exist in a neighbourhood of X_0 and are continuous at X_0 then show that f is differentiable at X_0 .

10. Let $f(x, y) = |x| + |y|$, for $(x, y) \in \mathbb{R}^2$. Show that f is continuous at $(0, 0)$ and no directional derivative of f at $(0, 0)$ exists.

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11. Let $f(x, y) = \sqrt{|xy|}$ for all $(x, y) \in \mathbb{R}^2$ and $(u, v) \in \mathbb{R}^2$ be such that $\|(u, v)\| = 1$. Show that directional derivative of f at $(0, 0)$ in the direction (u, v) exists if and only if $(u, v) = (1, 0)$ or $(u, v) = (0, 1)$.
12. Let $f(x, y) = \frac{x^2y}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(x, y) = 0$ for $(x, y) = (0, 0)$. Show that the directional derivative of f at $(0, 0)$ exist in all directions but f is not differentiable at $(0, 0)$.
13. Consider the function $f(x, y) = \frac{3x^2y-y^3}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(x, y) = 0$ for $(x, y) = (0, 0)$. Find the directional derivative of f at $(0, 0)$ in the direction $\frac{1}{\sqrt{2}}(1, 1)$. Discuss the differentiability of f at $(0, 0)$.
14. (a) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $(u, v) \in \mathbb{R}^2$ be such that $\|(u, v)\| = 1$. For $(x_0, y_0) \in \mathbb{R}^2$, show that $D_{(x_0, y_0)}f(u, v)$ is the derivative of $f(x_0 + tu, y_0 + tv)$ with respect to t at $t = 0$.
- (b) If $f(x, y) = xy$, using (a), find $D_{(1,1)}f(\frac{\sqrt{3}}{2}, \frac{1}{2})$.
15. Let $f(x, y) = x^2e^y + \cos(xy)$. Find the directional derivative of f at $(1, 2)$ in the direction $(\frac{3}{5}, \frac{4}{5})$.
16. Let $f(x, y) = 2x^2 + xy + y^2$ describes the temperature at (x, y) . Suppose a bug is at $(1, 1)$ and it decides to cool off. What is the best direction for it to move ?
17. For $X \in \mathbb{R}^3$, define $f(X) = \|X\|$. Let $X_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ and $\|X_0\| = 1$,
- (a) Show that $\nabla f(X_0) = X_0$. (b) Find the unit normal to the sphere $f(x, y, z) = 1$ at X_0 (c) Find the equation of the tangent plane of the sphere $f(x, y, z) = 1$ at X_0 .
18. Let $z = f(x, y)$, $x = r \cos(\theta)$, $y = r \sin(\theta)$.
- (a) Show that $\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x} \cos(\theta) + \frac{\partial f}{\partial y} \sin(\theta)$ & $\frac{1}{r} \frac{\partial z}{\partial \theta} = -\frac{\partial f}{\partial x} \sin(\theta) + \frac{\partial f}{\partial y} \cos(\theta)$
- (b) If $f(x, y) = x^2 + 2xy$, show that $\frac{\partial z}{\partial \theta} = 2(x^2 - xy - y^2)$.
19. If $z = f(x, y)$, $x = e^{2u} + e^{-2v}$ and $y = e^{-2u} + e^{2v}$ then, prove that $\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2 \left(x \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial x} \right)$.
20. If $J = \frac{\partial(x, y)}{\partial(u, v)}$ and $J^* = \frac{\partial(u, v)}{\partial(x, y)}$. Then
- (a) Prove that $JJ^* = 1$.
- (b) If $x = e^u \cos v$ and $y = e^u \sin v$, then verify $JJ^* = 1$.

Multiple Integrals

1. Let $R = [a, b] \times [c, d]$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = p(x)q(y)$ where $p : [a, b] \rightarrow \mathbb{R}$ and $q : [c, d] \rightarrow \mathbb{R}$ are continuous. Show that $\int \int_R f(x, y) \, dx dy = \left(\int_a^b p(x) dx \right) \left(\int_c^d q(y) dy \right)$.
2. Let $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$. Evaluate $\int \int_R \sin(x) \cos(y) \, dx dy$.
3. Evaluate $\int \int_R \cos(x^3) \, dx dy$ where R is the region in \mathbb{R}^2 bounded by $y = 3x^2$, $y = 0$ and $x = 1$.

4. Let R be the region lying below the curve $y = \cos x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and above the x-axis. Evaluate $\int \int_R \sin(x) \, dx dy$.
5. Let R be the region in \mathbb{R}^2 bounded by the curves $y = 2x^2$ and $y = 1 + x^2$. Evaluate $\int \int_R (2x^2 + y) \, dx dy$.
6. Evaluate $\int \int_R x \cos(y - \frac{y^3}{3}) \, dx dy$. Where $R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$.
7. Evaluate the following iterated integrals by interchanging the order of integration.

$$(a) \int_0^1 \int_y^1 \cos(x^2) \, dx dy \quad (b) \int_0^1 \int_{\sqrt{x}}^1 e^{y^3} \, dy dx \quad (c) \int_0^1 \int_{x^2}^1 x^3 e^{y^3} \, dy dx \quad (d) \int_0^1 \int_y^1 \frac{1}{1+x^4} \, dx dy$$

8. Evaluate $\int_0^1 (\arctan \pi x - \arctan x) \, dx$
9. Find the volume of the solid enclosed by the surfaces $z = 6 - x^2 - y^2$, $z = 2x^2 + y^2 - 1$, $x = -1$, $x = 1$, $y = -1$ and $y = 1$.
10. Let D be the solid bounded by the surfaces $y = x^2$, $y = 3x$, $z = 0$ and $z = x^2 + y^2$. Find the volume of D .
11. Let D be the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y + z = 1$ and $z = 0$. Find the volume of D .
12. Find the volume of the solid which is common to the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.
13. Consider the transformation $T : [0, 2\pi] \times [0, 1] \rightarrow \mathbb{R}^2$ given by $T(u, v) = (2v \cos u, v \sin u)$.
- (a) For a fixed $v_0 \in [0, 1]$, describe the set $\{T(u, v_0) : u \in [0, 2\pi]\}$.
- (b) Describe the set $\{T(u, v) : (u, v) \in [0, 2\pi] \times [0, 1]\}$.
14. Let R be the region in \mathbb{R}^2 bounded by the straight lines $y = x$, $y = 3x$ and $x + y = 4$. Consider the transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(u, v) = (u - v, u + v)$. Find the set $S \subset \mathbb{R}^2$ satisfying $T(S) = R$.
15. Let R be the region in \mathbb{R}^2 bounded by the curve defined in the polar co-ordinates $r = 1 - \cos \theta$, $0 \leq \theta \leq \pi$ and the x-axis. Consider the transformation $T : [0, \pi] \times [0, 1] \rightarrow \mathbb{R}^2$ defined by $T(r, \theta) = (r \cos \theta, r \sin \theta)$. Let S be the subset of $[0, \pi] \times [0, 1]$ satisfying $T(S) = R$. Sketch the region S in R .
16. Using the change of variable $u = x + y$ and $v = x - y$, show that

$$\int_0^1 \int_0^x (x - y) \, dy dx = \int_0^1 \int_v^{2-v} \frac{v}{2} \, dudv.$$

17. Let R be the region bounded by $x = 0$, $x = 1$, $y = x$ and $y = x + 1$. Show that

$$\int \int_R \frac{1}{\sqrt{xy - x^2}} \, dx dy = \left(\int_0^1 \frac{1}{\sqrt{u}} du \right) \left(\int_0^1 \frac{1}{\sqrt{v}} dv \right).$$

18. Evaluate $\iint_R (x^2 + y^2) \, dx dy$ where R is the region bounded by $x^2 - y^2 = a$, $x^2 - y^2 = b$, $2xy = c$ and $2xy = d$. $0 < a < b$, $0 < c < d$. (Hint: Use $J.J^* = 1$)
19. Using suitable change of variables, evaluate $\iint_R \sqrt{(x^2 + y^2)} \, dx dy$ where R is the region bounded by $x = 0$, $x = 1$, $y = 0$ and $y = x$.
20. Let D denotes the solid bounded by the surfaces $y = x$, $y = x^2$, $z = x$ and $z = 0$. Evaluate $\iiint_D y \, dx dy dz$.
21. Let D denote the solid bounded below by the planes $z + y = 2$, above by the cylinder $z + y^2 = 4$ and on the sides $x = 0$ and $x = 2$. Evaluate $\iiint_D x \, dx dy dz$.
22. Suppose $\int_0^4 \int_{\sqrt{x}}^2 \int_0^{2-y} dz dy dx = \iiint_D dx dy dz$ for some region $D \subset \mathbb{R}^3$.
- (a) Sketch the region D . (b) Sketch the projections of D on the xy , yz and xz planes. (c) Write $\int_0^4 \int_{\sqrt{x}}^2 \int_0^{2-y} dz dy dx$ as iterated integrals of other orders.
23. Let $D = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} \leq 1 \right\}$ and $E = \{ (u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 \leq 1 \}$. Show that $\iiint_D dx dy dz = \iiint_E 24 \, du dv dw$.
24. In each of the following cases, describe the solid D in terms of the cylindrical co-ordinates.
- (a) Let D be the solid that is bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.
- (b) Let D be the solid that lies within the cylinder $x^2 + (y - 1)^2 = 1$ below the paraboloid $z = x^2 + y^2$ and above the plane $z = 0$.
- (c) Let S denote the torus generated by revolving the circle $\{ (x, z) : (x - 2)^2 + z^2 = 1 \}$ about the z -axis. Let D be the solid that is bounded by the surface S and below by $z = 0$.
25. Let D be the solid that lies inside the cylinder $x^2 + y^2 = 1$, below the cone $z = \sqrt{4(x^2 + y^2)}$ and above the plane $z = 0$. Evaluate $\iiint_D x^2 \, dx dy dz$.
26. Evaluate $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^4 x \, dz dy dx$.
27. Describe the following regions in terms of the spherical coordinates.
- (a) The regions that lies inside the sphere $x^2 + y^2 + (z - 2)^2 = 4$ and outside the sphere $x^2 + y^2 + z^2 = 1$.
- (b) The region that lies below the sphere $x^2 + y^2 + z^2 = z$ and above the cone $z = \sqrt{x^2 + y^2}$.
- (c) The region that is enclosed by the cone $z = \sqrt{3(x^2 + y^2)}$ and the planes $z = 1$ and $z = 2$.
28. Let D denote the solid bounded above by the planes $z = 4$ and below by the cone $z = \sqrt{(x^2 + y^2)}$. Evaluate $\iiint_D \sqrt{(x^2 + y^2 + z^2)} \, dx dy dz$.
29. Let D denote the solid enclosed by the spheres $x^2 + y^2 + (z - 1)^2 = 1$ and $x^2 + y^2 + z^2 = 3$. Using spherical coordinates, set up iterated integrals that gives the volume of D .