

Multiple Integrals

① Let $f: R \rightarrow \mathbb{R}$, define $f(x, y) = p(x)q(y)$
 $[R = [a, b] \times [c, d]]$ [Use Fubini's theorem]

$$\iint_R f(x, y) dx dy = \int_c^d \left(\int_a^b p(x)q(y) dx \right) dy$$

$$= \int_c^d A(x)q(y) dy = A(x) \int_c^d q(y) dy$$

where $A(x) = \int_a^b p(x) dx = A(x)A(y)$

$$A(y) = \int_c^d q(y) dy$$

$$\therefore \iint_R f(x, y) = A(x)A(y).$$

② Using Fubini's theorem, (or use Problem 1).

$$\iint_R \sin x \cos y dx dy = \int_0^{\pi/2} \sin x dx \int_0^{\pi/2} \cos y dy = 1.$$

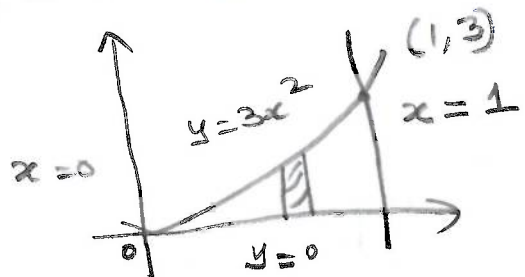
③ R is bounded by $y = 3x^2$, $y = 0$, $x = 0$ and $x = 1$

$$\iint_R \frac{\cos(x^3)}{3x^2} dx dy$$

$$= \int_{x=0}^1 \int_{y=0}^{3x^2} \cos(x^3) dy dx$$

$$= \int_{x=0}^1 \cos(x^3) y \Big|_0^{3x^2} dx = \int_{x=0}^1 3x^2 \cos(x^3) dx = \sin 1$$

(Hint: Let $x^3 = t$).

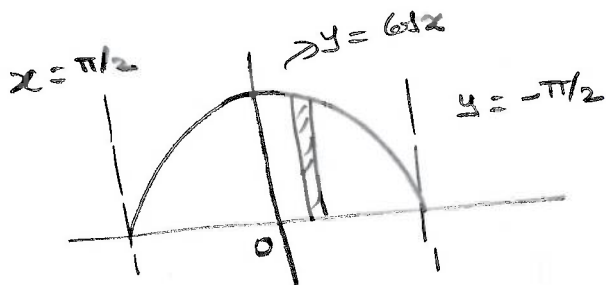


④

$$\iint_R \sin x \cos x dx dy$$

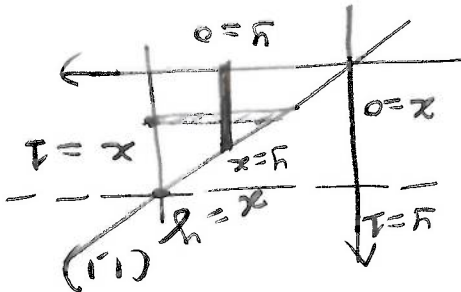
$$= \int_{-\pi/2}^{\pi/2} \int_{y=0}^{\cos x} \sin x dy dx$$

$$= \int_{-\pi/2}^{\pi/2} \sin x \cos x dx = \frac{\sin^2 x}{2} \Big|_{-\pi/2}^{\pi/2} = 0.$$



(2)

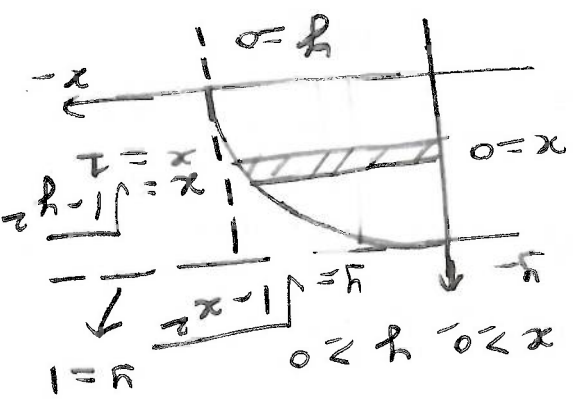
$$= \int_{x=0}^1 \int_{y=0}^x \cos(x^2) \cdot x \, dy \, dx = \int_0^1 x \cos(x^2) dx = \frac{1}{2} \sin(1)$$



(*) $\int_1^x \int_1^y \cos(x^2) dx dy$

[Hint: Let $y - \frac{y^3}{3} = t \Rightarrow 1 - \frac{y^3}{3} = t \Rightarrow y = \sqrt[3]{3(1-t)}$

$$= \int_0^{\frac{2}{3}} \int_{\frac{2}{3}-t}^{\frac{2}{3}} \cos(t) dt dy = \int_{\frac{2}{3}}^0 \cos(t) \cdot \frac{2}{3} dt = \frac{2}{3} [\sin(t)]_{\frac{2}{3}}^0 = \frac{2}{3} \sin\left(\frac{2}{3}\right)$$

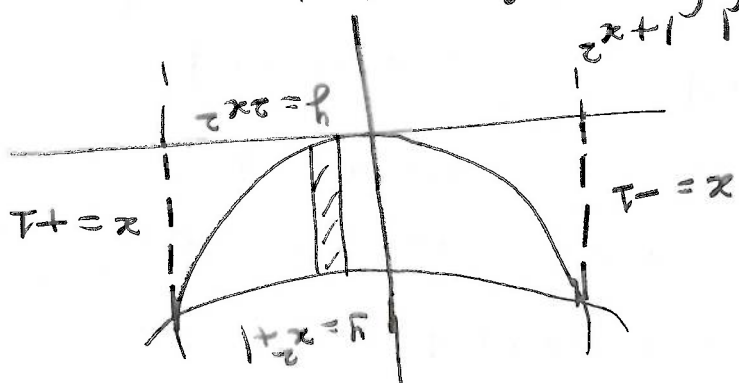


$$= \int_0^1 \int_0^{\sqrt{1-y^2}} x \cos\left(y - \frac{y^3}{3}\right) dx dy$$

(3)

$R = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$

$$= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2x^2 + y) dx dy$$



$2x^2 = 1 + x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

(5)

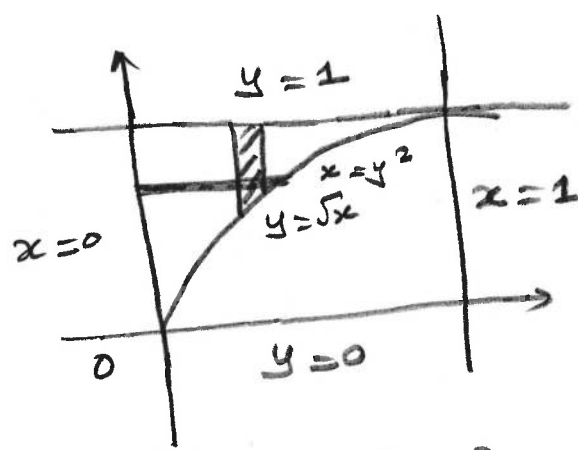
m solving

$$\textcircled{b} \int_{x=0}^1 \int_{y=\sqrt{x}}^1 e^{y^3} dy dx$$

$$y = \sqrt{x} \Rightarrow y^2 = x$$

$$= \int_{y=0}^1 \left(\int_{x=0}^{y^2} e^{y^3} dx \right) dy$$

$$= \int_0^1 y^2 e^{y^3} dy = \frac{1}{3} (e-1)$$

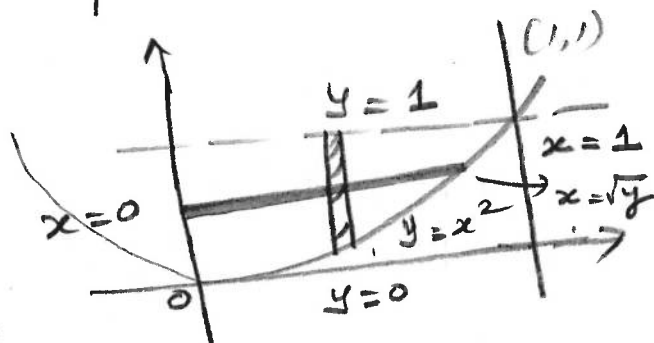


Hint: Let $t = y^3$

$$\textcircled{c} \int_{x=0}^1 \int_{y=x^2}^1 x^3 e^{y^3} dy dx$$

$$= \int_{y=0}^1 \left(\int_{x=0}^{\sqrt[4]{y}} x^3 e^{y^3} dx \right) dy$$

$$= \int_{y=0}^1 e^{y^3} \left. \frac{x^4}{4} \right|_{x=0}^{\sqrt[4]{y}} dy = \int_0^1 e^{y^3} \times \frac{y^2}{4} dy = \frac{1}{12} (e-1)$$



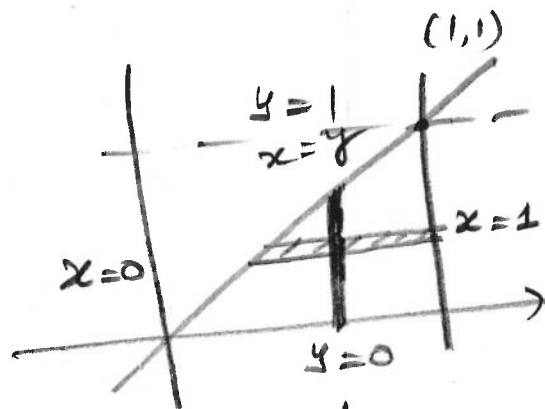
$$\textcircled{d} \int_{y=0}^1 \int_{x=y}^1 \frac{1}{1+x^4} dx dy$$

$$= \int_{x=0}^1 \left(\int_{y=0}^x \frac{1}{1+x^4} dy \right) dx$$

$$= \int_{x=0}^1 \frac{1}{1+x^4} \times y \Big|_0^x dx = \int_0^1 \frac{x}{1+x^4} dx = \frac{1}{2} \int_0^1 \frac{dt}{1+t^2}$$

$$\left(\text{Let } x^2 = t \Rightarrow 2x dx = dt \right)$$

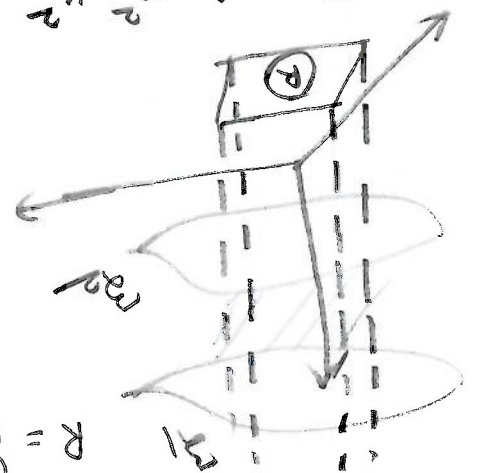
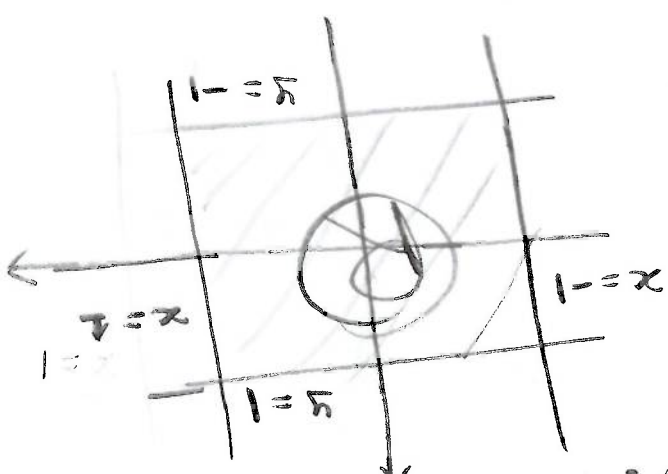
$$= \frac{\pi}{8}$$



(4)

Volume = $\int_{-1}^1 \int_{-1}^1 [7 - (3x^2 + 2y^2)] dx dy$

clearly $z_1 - z_2 = 6 - x^2 - y^2 - (2x^2 + y^2 - 1) = 6 - 3x^2 - 2y^2 + 1 = 7 - (3x^2 + 2y^2) \geq 0$ for all $(x, y) \in R$.



$R = \{(x, y) : (x, y) \in [-1, 1] \times [-1, 1]\}$.

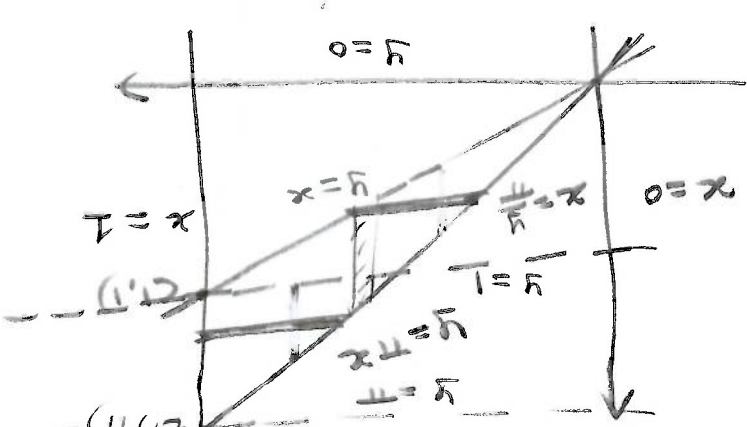
(9)

$\therefore R = R_1 \cup R_2$

$\int \int_R \frac{1}{1+y^2} dy dx = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_0^1 \frac{1}{1+y^2} dy dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_1^{\pi} \frac{1}{1+y^2} dy dx$

$R_2 = \{(x, y) : 1 \leq y \leq \pi, \frac{\pi}{4} \leq x \leq 1\}$

Let $R_1 = \{(x, y) : 0 \leq y \leq 1, \frac{\pi}{4} \leq x \leq y\}$



$\int_0^{\frac{\pi}{4}} (\tan^{-1} \pi x - \tan^{-1} x) dx = \int_0^{\frac{\pi}{4}} \frac{1}{1+y^2} dy dx$

(8)

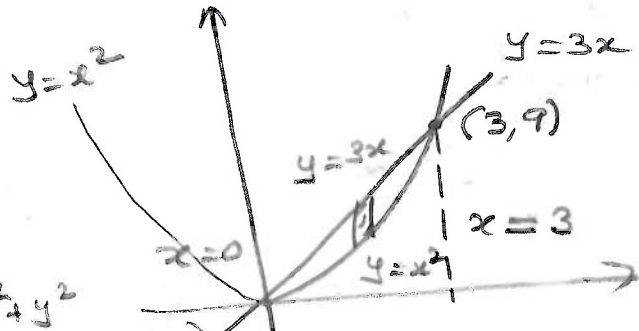
10) on solving

$$x^2 = 3x$$

$$\Rightarrow x^2 - 3x = 0$$

$$x(x-3) = 0$$

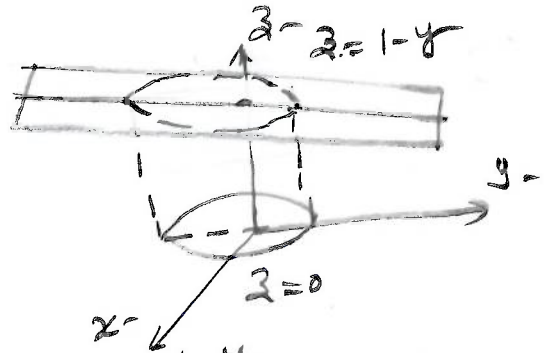
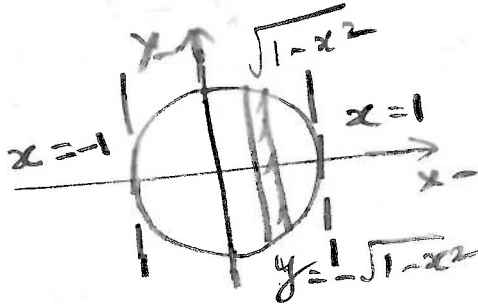
$$x = 0 \text{ and } x = 3$$



$$\text{Volume} = \int_{x=0}^3 \left(\int_{y=x^2}^{3x} (x^2 + y^2) dz \right) dy dx$$

$$= \int_{x=0}^3 \left(\int_{y=x^2}^{3x} (x^2 + y^2) dy \right) dx$$

11)



$$\text{Volume} = \int_{-1}^1 \left(\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-y) dz \right) dy dx$$

$$= \int_{-1}^1 \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-y) dy dx \quad (\text{since } z = 1-y)$$

$$= \int_{-1}^1 \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy dx - \int_{-1}^1 \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y dy dx$$

$$= \text{Area of circle} + 0 \quad (\text{since the integrand is odd w.r.t } y)$$

$$= \pi \times 1^2 = \underline{\underline{\pi}}$$

$x^2 + y^2 = 1, x^2 + z^2 = 1.$

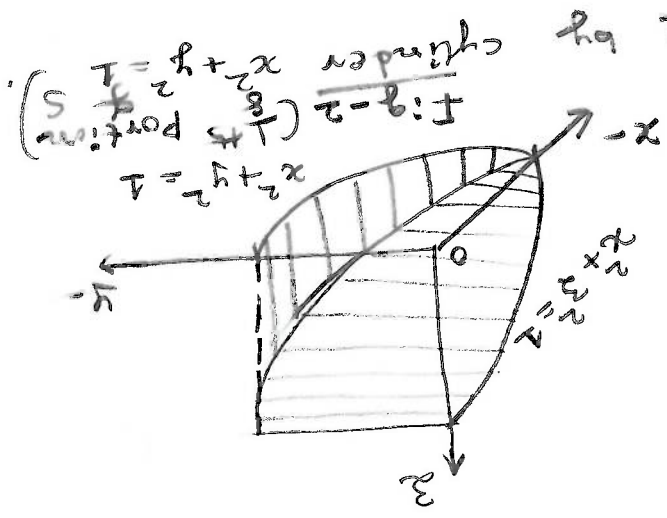
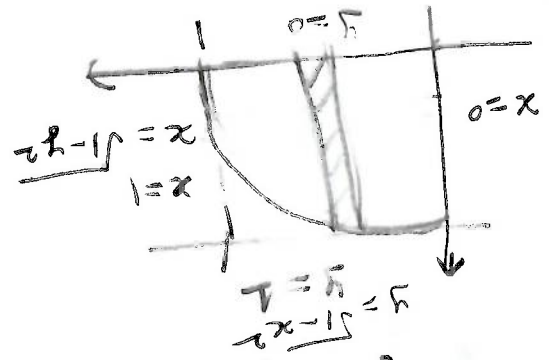


Fig-2 (1/8 portion of cylinder $x^2 + z^2 = 1$)

The solid is enclosed by

Fig-1 and the surfaces

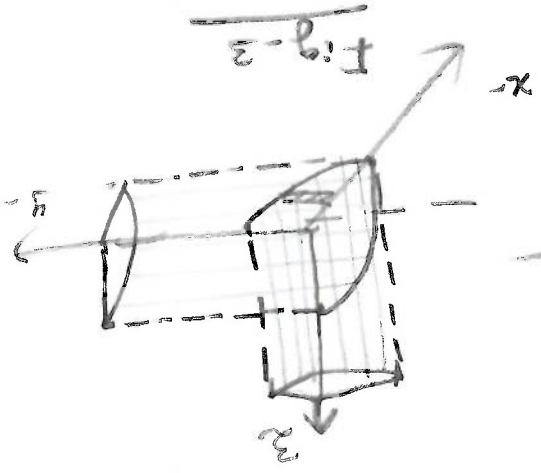
$z = \sqrt{1-x^2}$ and $z = -\sqrt{1-x^2}$.

Let $R = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

$$\begin{aligned} \therefore V &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz dy dx \\ &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2} dy dx \\ &= 8 \int_{-1}^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2} dy dx \end{aligned}$$

$= \frac{16}{3}$

Fig-3 shows the 1/8 of the required volume. You can also refer to Fig-2, for the clarity.

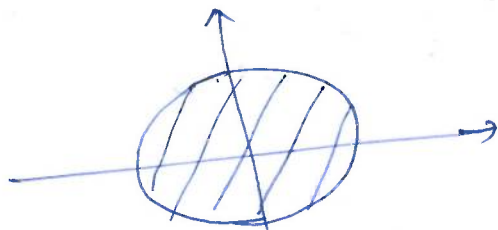
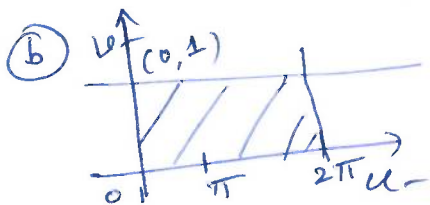


13) @ If $x = 2r_0 \cos u$ and $y = r_0 \sin u$

$\Rightarrow \frac{x}{2r_0} = \cos u \quad \frac{y}{r_0} = \sin u$

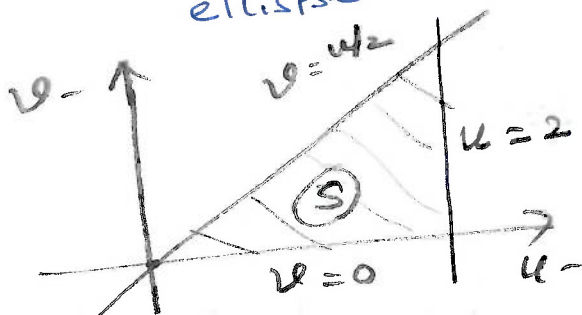
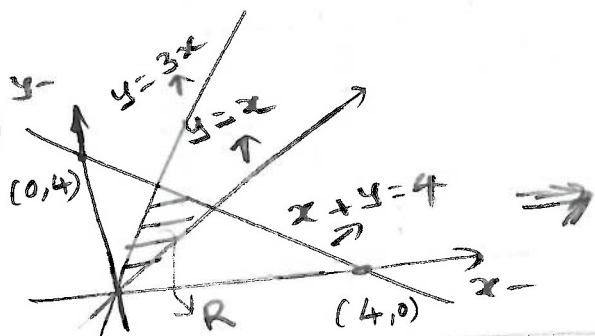
$\frac{x^2}{4r_0^2} + \frac{y^2}{r_0^2} = 1 \Rightarrow \frac{x^2}{4} + \frac{y^2}{1} = r_0^2$

\therefore The set $\{T(u, r_0) : u \in [0, 2\pi]\}$ is an ellipse.



ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$.

14



Let $x = u - v$
 $y = u + v$
 $\therefore x + y = 2u$

$u = \frac{1}{2}(x + y)$

$v = y - u = y - \frac{x}{2} - \frac{y}{2} = \frac{y}{2} - \frac{x}{2} = \frac{1}{2}(y - x)$

$v = \frac{1}{2}(y - x)$

Case (i): if $y = x \Rightarrow v = 0$

Case (ii): if $y = 3x \Rightarrow v = \frac{1}{2}(3x - x) = \frac{2x}{2} = x$

and $u = \frac{1}{2}(x + 3x) = \frac{1}{2} \times 4x = 2x$
 $(\Rightarrow x = u/2)$

$\Rightarrow v = x = \frac{u}{2}$

$\Rightarrow v = u/2$

Case (iii): if $x + y = 4 \Rightarrow u = \frac{1}{2} \times 4 \Rightarrow u = 2$

(8)

$\int_0^a \int_0^x (x-y) dy dx = \int_0^a \left[xy - \frac{y^2}{2} \right]_0^x dx = \int_0^a \left(x^2 - \frac{x^2}{2} \right) dx = \int_0^a \frac{x^2}{2} dx = \frac{1}{2} \left[\frac{x^3}{3} \right]_0^a = \frac{1}{6} a^3$

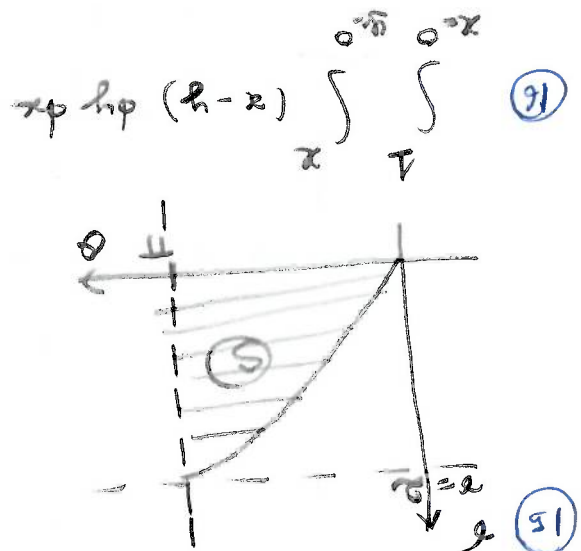
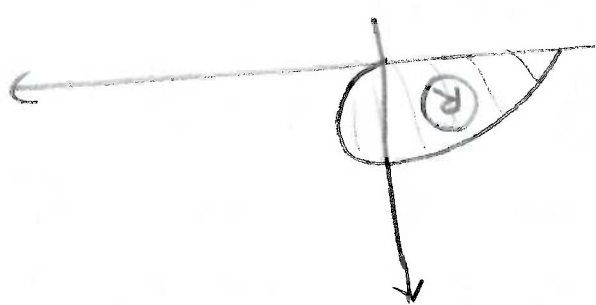
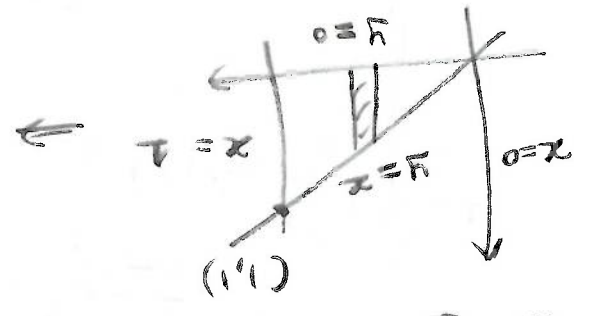
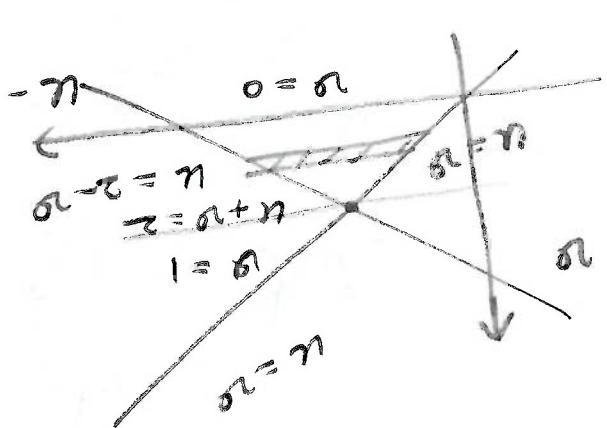
$\frac{\partial}{\partial x} \int_0^x (x-y) dy = x - \frac{x}{2} = \frac{x}{2}$

$\frac{\partial}{\partial x} \left(\frac{1}{6} a^3 \right) = \frac{1}{2} a^2$

$\frac{\partial}{\partial x} \left(\frac{1}{6} a^3 \right) = \frac{1}{2} a^2 \Rightarrow \frac{\partial}{\partial x} \left(\frac{1}{6} a^3 \right) = \frac{\partial}{\partial x} \left(\frac{1}{2} a^2 \right)$

$\frac{\partial}{\partial x} \left(\frac{1}{6} a^3 \right) = \frac{1}{2} a^2 \Rightarrow \frac{\partial}{\partial x} \left(\frac{1}{6} a^3 \right) = \frac{\partial}{\partial x} \left(\frac{1}{2} a^2 \right)$

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17) Take $u=x$ and $v=1-x$.

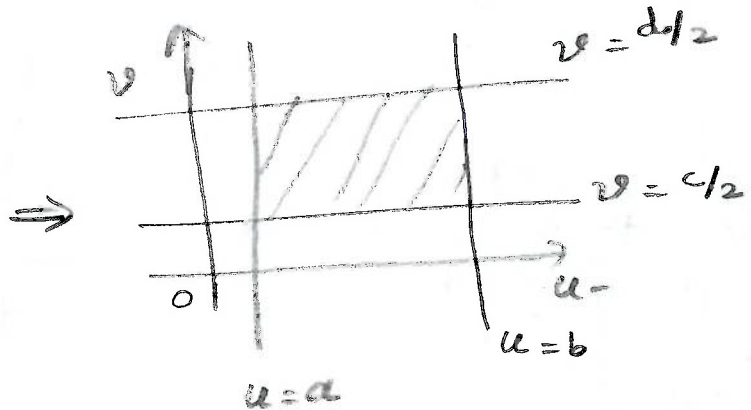
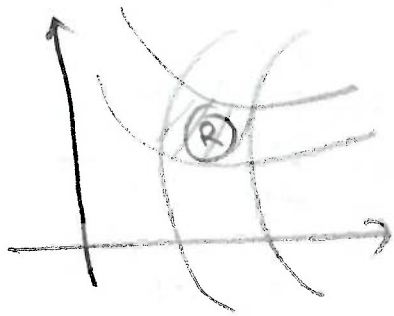
Then $y=x \Rightarrow v=0$ and $y=x+1 \Rightarrow v=1$.

$$\therefore \iint_R \frac{dx dy}{\sqrt{xy-x^2}} = \iint_S \frac{1}{\sqrt{uv}} |J| du dv$$

where S is the region in the uv -plane bounded by the lines

$u=0, u=1, v=0$ and $v=1$.

18)



Let $x^2 - y^2 = u : xy = v$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & y \\ -2y & x \end{vmatrix} = 2(x^2 + y^2); \quad \left(\begin{array}{l} \text{we know that} \\ J J^* = 1 \end{array} \right)$$

$\neq 0$ in \mathbb{R}

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2(x^2 + y^2)}$$

$$\iint_R (x^2 + y^2) dx dy = \int_{u=a}^b \int_{v=c/2}^{d/2} (x^2 + y^2) \times \frac{1}{2(x^2 + y^2)} du dv$$

$$= \frac{1}{2} \int_a^b du \times \int_{c/2}^{d/2} dv$$

$$= \frac{1}{2} (b-a) \left(\frac{d}{2} - \frac{c}{2} \right)$$

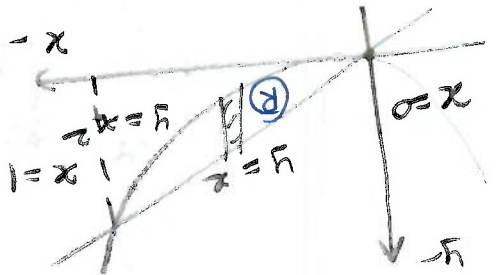
$$= \frac{1}{4} (b-a)(d-c).$$

9

10

$$\iiint_D y \, dx \, dy \, dz = \int_0^1 \int_x^1 \int_x^1 y \, dz \, dy \, dx = \int_0^1 \int_x^1 y \, dy \, dx = \int_0^1 \left[\frac{y^2}{2} \right]_x^1 dx = \int_0^1 \left(\frac{1}{2} - \frac{x^2}{2} \right) dx = \left[\frac{x}{2} - \frac{x^3}{6} \right]_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

The projection of the solid D on xy -plane is given by $R = \{ (x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, x^2 \leq y \leq x \}$. The solid D lies above the surface $z = f_1(x,y) = 0$ and below $z = f_2(x,y) = x$.



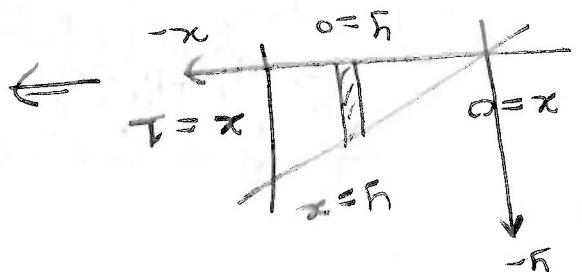
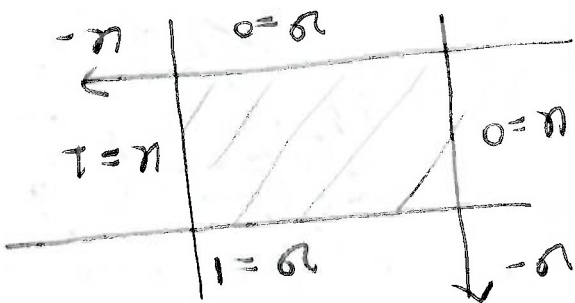
20

if $x=0$: $u=0$
 if $x=1$: $u=1$
 if $y=x$: $u = \sqrt{v} \Rightarrow v = u^2$
 if $y=0$: $u^2 = 0 \Rightarrow u=0 \Rightarrow v=0$

$$J = \begin{vmatrix} 1 & 2u \\ 0 & u \end{vmatrix} = u$$

$$x^2 + y^2 = u^2 + u^2 = u^2(1+u^2)$$

$$\int_0^1 \int_{u^2}^1 \sqrt{x^2 + y^2} \, dx \, dy = \int_0^1 \int_{u^2}^1 \sqrt{u^2(1+u^2)} \, dx \, du = \int_0^1 \left[\frac{x}{2} \sqrt{1+u^2} + \frac{1}{2} \sin^{-1}\left(\frac{x}{u}\right) \right]_{u^2}^1 du$$



19

② $z + y = 2; z + y^2 = 4$

$\Rightarrow z = 2 - y; z = 4 - y^2$

$2 - y = 4 - y^2$

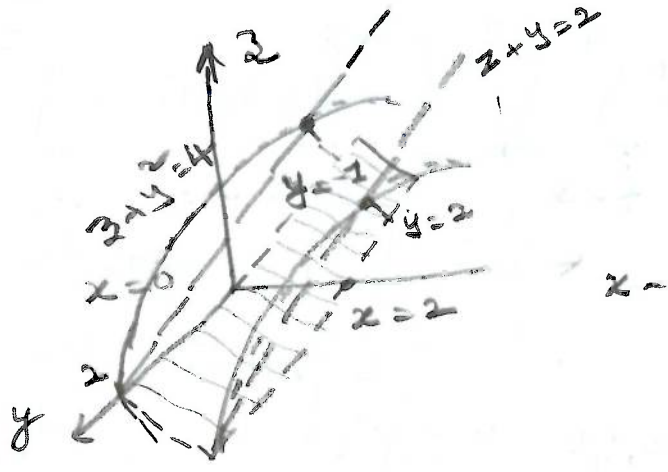
$\Rightarrow y^2 - y - 2 = 0$

$y^2 - 2y + y - 2 = 0$

$y(y - 2) + 1(y - 2) = 0$

$(y + 1)(y - 2) = 0$

$y = 2 \text{ and } y = -1.$



The projection of the solid D on the xy-plane is given by $R = [0, 2] \times [-1, 2]$

The solid lies above

$z = f_1(x, y) = 2 - y$ and below

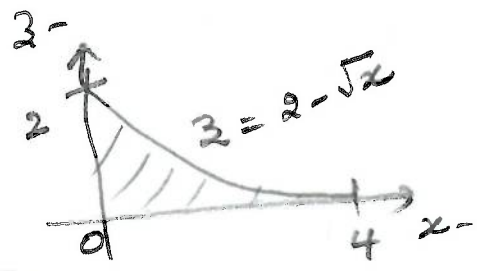
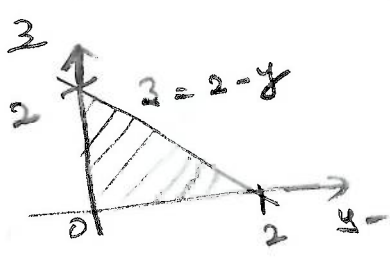
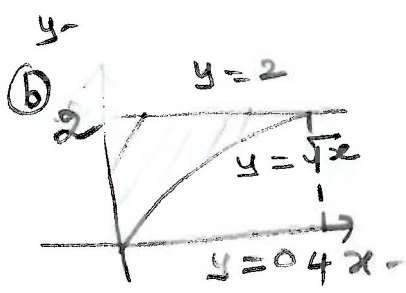
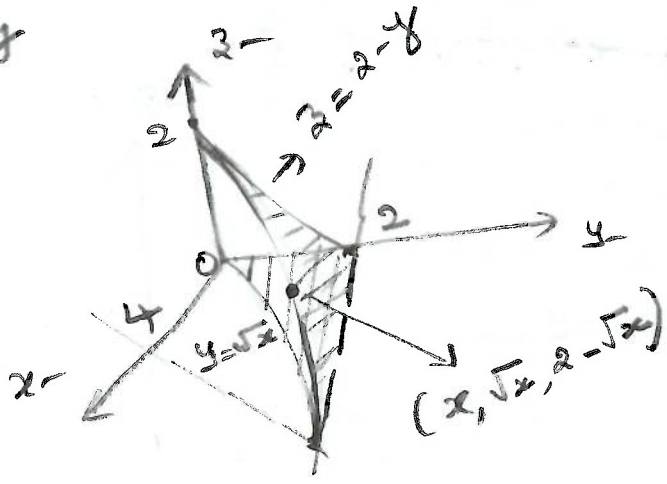
$z = f_2(x, y) = 4 - y^2$

$\therefore \iiint_D x \, dz \, dy \, dx = \iint_R \left(\int_{z=2-y}^{z=4-y^2} x \, dz \right) dy \, dx$

$= \int_0^2 \int_{-1}^2 \int_{2-y}^{4-y^2} x \, dz \, dy \, dx$

② $\int_{x=0}^4 \int_{y=\sqrt{x}}^2 \int_{z=0}^{2-y}$

②

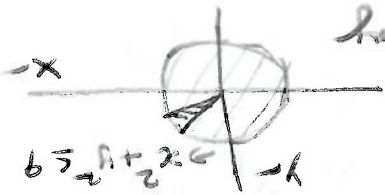


①①

$$\therefore D = \left\{ (r, \theta, z) : 0 \leq \theta \leq 2\pi, 0 \leq r \leq 3, r^2 \leq z \leq 36-3r^2 \right\}$$

$$z = x^2 + y^2 = r^2 \text{ and } z = 36 - 3(x^2 + y^2) = 36 - 3r^2$$

The solid is bounded by

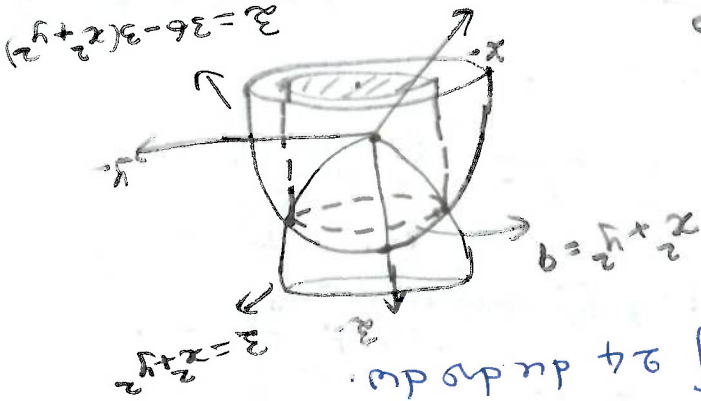


on the xy -plane is the circular disk $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$

The projection of the solid D

$$\Rightarrow x^2 + y^2 = 9$$

② Solving $x^2 + y^2 = 36 - 3(x^2 + y^2)$



$$\therefore \iiint_D dx dy dz = \iiint_D 24 \, du dv dw$$

$$\text{So } \iiint_D dx dy dz = \int \int \int |J| \, du dv dz = 24 \int \int \int du dv dz$$

③ Let $x = 2u, y = 4v$ and $z = 3w$

$$\therefore J = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 24$$

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = u^2 + v^2 + w^2 \leq 1$$

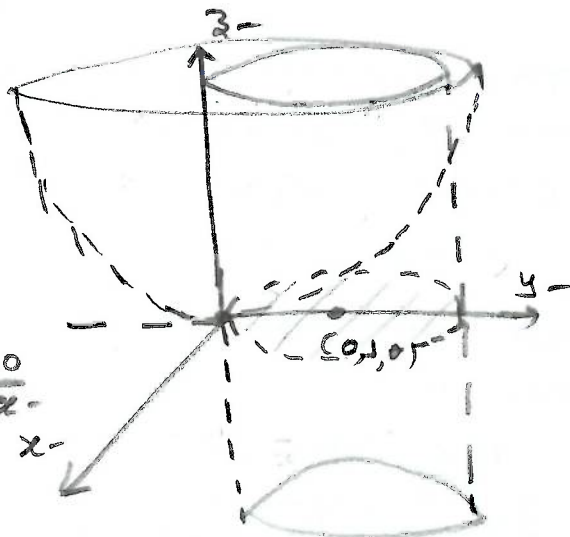
$$\int_0^{\sqrt{x}} \int_{-y}^y \int_{x-3}^{x-3} dx dy dz = \int_0^{\sqrt{x}} \int_{-y}^y (x-3) dy dx$$

$$= \int_0^{\sqrt{x}} \int_{-y}^y (x-3) dy dx = \int_0^{\sqrt{x}} (x-3) \int_{-y}^y dy dx = \int_0^{\sqrt{x}} (x-3) \cdot 2y dx$$

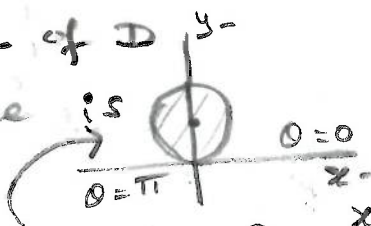
④ $\int_0^{\sqrt{x}} \int_{-y}^y \int_{x-3}^{x-3} dz dy dx = \int_0^{\sqrt{x}} \int_{-y}^y (x-3) dy dx$

24 (b)

$x^2 + (y-1)^2 = 1$; $z = x^2 + y^2$
 above $z=0$ (xy-plane)



The projection of D on the xy -plane is given by



$\{(x, y) \in \mathbb{R}^2 : x^2 + (y-1)^2 \leq 1\}$

which is described in cylindrical co-ordinates as $\{(r, \theta) : 0 \leq \theta \leq \pi, 0 \leq r \leq 2 \sin \theta\}$

$x = r \cos \theta, y = r \sin \theta \Rightarrow x^2 + y^2 = r^2$ | $r=0$
 $x^2 + y^2 - 2y + 1 = 1 \Rightarrow r^2 - 2r \sin \theta = 0 \Rightarrow r(r - 2 \sin \theta) = 0$ | $r=2 \sin \theta$

$\therefore D = \{(r, \theta, z) : 0 \leq \theta \leq \pi, 0 \leq r \leq 2 \sin \theta, 0 \leq z \leq r^2\}$

(c) The projections of the solid D on the xy -plane is the region between the circles $r=1$ and $r=3$. Also $0 \leq \theta \leq 2\pi$.

Consider the cross section of the solid, \perp to xy -plane, corresponding to a fixed θ .

The cross section is a circle which is shown in Fig-2.

The equation of the circle can be considered as

$(r-2)^2 + z^2 = 1$ for $1 \leq r \leq 3$.

$\therefore D = \{(r, \theta, z) : 0 \leq \theta \leq 2\pi, 1 \leq r \leq 3, 0 \leq z \leq \sqrt{1 - (r-2)^2}\}$

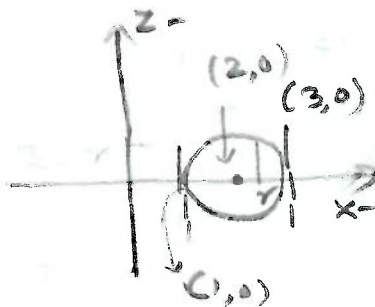
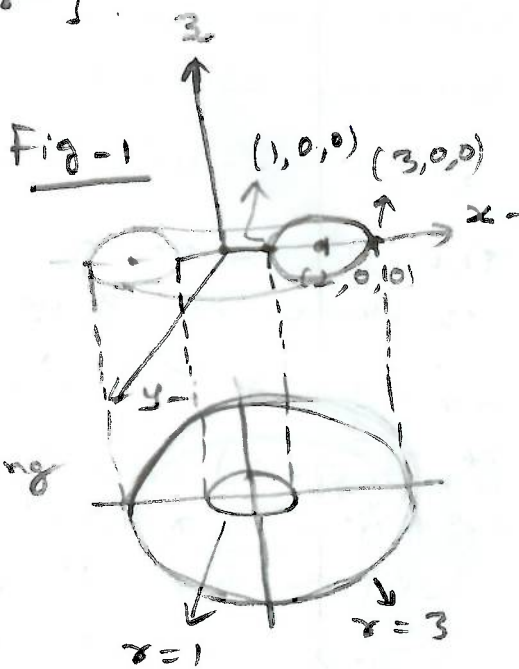


Fig. 2

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The projection of the solid D on the xy -plane is the circular disk $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

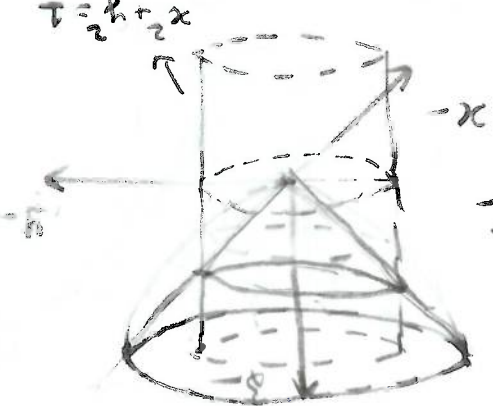


We will use the cylindrical co-ordinates.

The solid D is bounded by $z=0$ and

$$z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - r^2} = 2r \quad 0 \leq r \leq 1$$

$$\therefore \iiint_D x^2 dx dy dz = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} r^2 \cos^2 \theta \times r dz dr d\theta$$



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Note that, $\sqrt{4-x^2}$

$$\int_{-2}^2 \int_{x^2+y^2}^{-\sqrt{4-x^2}} x^2+y^2$$

where the solid D is

bounded below by

$$z = x^2 + y^2 \text{ and above } z = 4$$

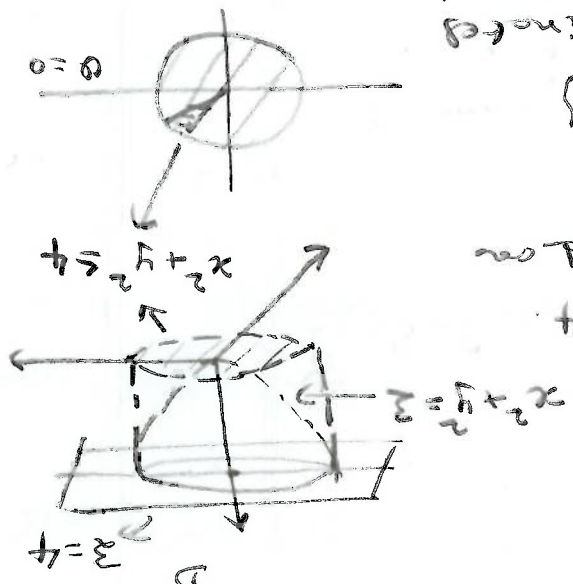
The projection of the solid on

xy -plane is given by

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$$

By the cylindrical co-ordinates

$$\iiint_D x dz dy dx = \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{4-r^2}} r^2 \cos \theta \times r dz dr d\theta$$



27) a) Let

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$$

$$\therefore x^2 + y^2 + z^2 = \rho^2$$

$$\text{So, } x^2 + y^2 + z^2 = 1 \Rightarrow \boxed{\rho = 1}$$

and

$$x^2 + y^2 + (z-2)^2 = 4$$

$$\Rightarrow x^2 + y^2 + z^2 - 4z + 4 = 4$$

$$\Rightarrow \rho^2 - 4\rho \cos \phi = 0$$

$$\Rightarrow \rho = 0 \text{ (or) } \boxed{\rho = 4 \cos \phi} \quad \text{--- (2)}$$

From (1) and (2),

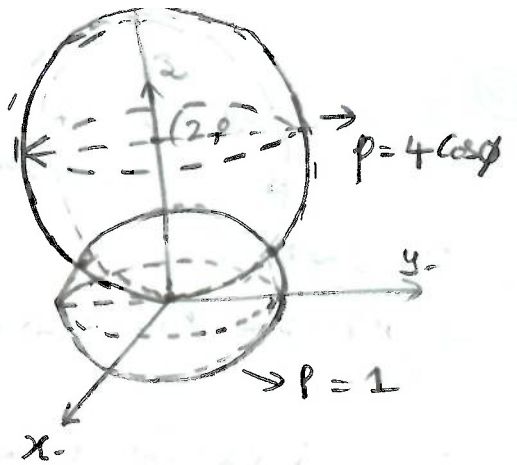
$$4 \cos \phi = 1 \Rightarrow \cos \phi = \frac{1}{4}$$

Two spheres intersect at $\cos \phi = \frac{1}{4}$.

For a fixed $\phi \in [0, \cos^{-1}(\frac{1}{4})]$,

ρ varies from 1 to $4 \cos \phi$ in the given region.

$$\therefore \left\{ (\rho, \theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \cos^{-1}\left(\frac{1}{4}\right), 0 \leq \rho \leq 4 \cos \phi \right\}$$



6) $x^2 + y^2 + z^2 = 2 \Rightarrow \rho^2 = \rho \cos \phi \Rightarrow \rho = \cos \phi$

$$z = \sqrt{x^2 + y^2}$$

$$\Rightarrow x^2 + y^2 - z^2 = 0$$

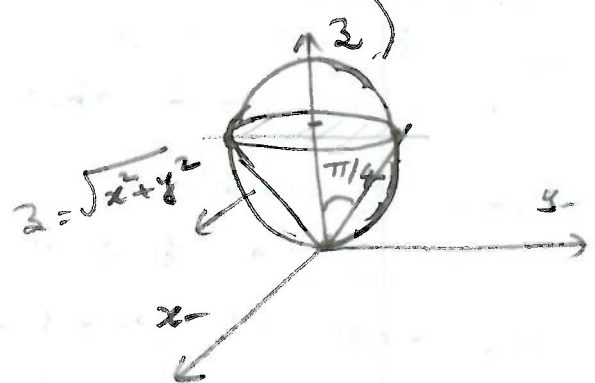
$$\rho^2 \sin^2 \phi = \rho^2 \cos^2 \phi$$

$$\sin \phi = \cos \phi \Rightarrow \phi = \frac{\pi}{4}$$

For a fixed $\phi \in [0, \pi/4]$,

ρ varies from 0 to $\cos \phi$ in the given region.

$$\therefore \left\{ (\rho, \theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{4}, 0 \leq \rho \leq \cos \phi \right\}$$



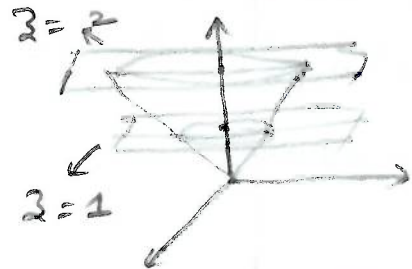
7) $z = \sqrt{3(x^2 + y^2)} = \sqrt{3(\rho^2 \sin^2 \phi)}$

$$\Rightarrow \rho \cos \phi = \sqrt{3} \rho \sin \phi \Rightarrow \phi = \frac{\pi}{6}$$

For a fixed $\phi \in [0, \pi/6]$,

ρ varies from $\sec \phi$ to $2 \sec \phi$

$$\therefore \left\{ (\rho, \theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{6}, \sec \phi \leq \rho \leq 2 \sec \phi \right\}$$



if $z = 1$
 $\Rightarrow 1 = \rho \cos \phi \Rightarrow \rho = \sec \phi$
 at $z = 2$
 $2 = \rho \cos \phi$
 $\Rightarrow \rho = 2 \sec \phi$

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$$+ \int_0^{\pi/6} \int_{\pi/2}^{2\pi} \int_0^{\sqrt{3}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\therefore \text{Volume} = \int_0^{\pi/6} \int_{\pi/2}^{2\pi} \int_0^{\sqrt{3}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

The required volume = volume of the portion of the cone that lies inside the sphere $x^2 + y^2 + z^2 \leq 3$ that lies inside the region $\phi = \pi/6$ + volume of the portion of the sphere that lies inside the sphere $x^2 + y^2 + z^2 \leq 3$.

$$\sqrt{3} = 2 \cos\phi \Rightarrow \phi = \pi/6$$

From ① and ②,

$$\boxed{p = 0 \text{ (or) } p = 2 \cos\phi} \text{--- ③}$$

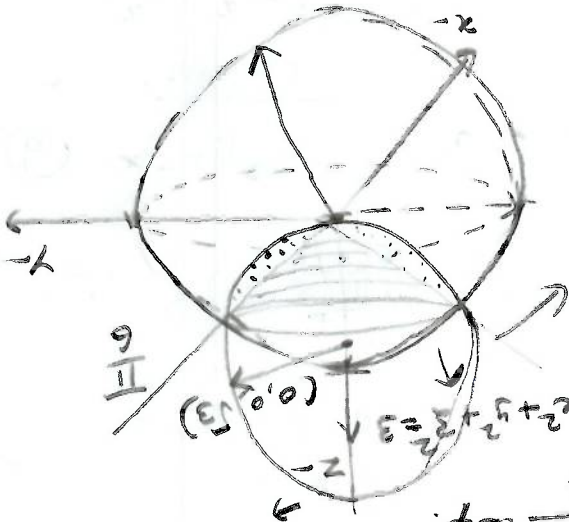
$$\Rightarrow p^2 - 2p \cos\phi = 0$$

$$x^2 + y^2 + z^2 - 2z = r^2$$

$$\Rightarrow p^2 = 3 \Rightarrow p = \sqrt{3} \text{--- ①}$$

$$x^2 + y^2 + z^2 = 3$$

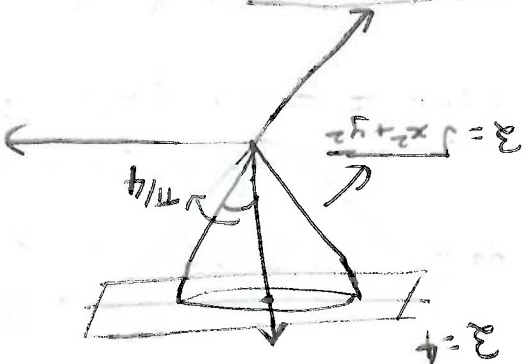
$$x^2 + y^2 + (z-1)^2 = 2$$



$$= 2\pi \int_0^{\pi/4} \int_0^3 \frac{\sin\phi \, d\phi}{\cos^4\phi} \, d\phi$$

$$= \int_0^{\pi/4} \int_{2\pi}^{2\pi} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\sqrt{x^2 + y^2 + z^2} = \rho$$



$$\Rightarrow \rho / \cos\phi = \rho \sin\phi \Rightarrow \phi = \pi/4$$

$$z = \sqrt{x^2 + y^2}$$

$$\Rightarrow \rho = 4 \sec\phi$$

$$z = 4 \Rightarrow \rho \cos\phi = 4$$

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