

Tutorial Sheet #1.

Example of a written solution.

Exercise 1

$$f) \text{ For } (x, y) \in \mathbb{R}^2, \quad f(x, y) = \begin{cases} \sin(x^2 + y^2) / x^2 + y^2 & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{otherwise.} \end{cases}$$

For $(x, y) \in \mathbb{R}^2$, $(x, y) \neq (0, 0)$ and $(x, y) \rightarrow (0, 0)$
we have $x^2 + y^2 \rightarrow 0$
 $(x, y) \rightarrow 0$

$$\text{and } \frac{\sin(t)}{t} \rightarrow 1 \quad t \rightarrow 0, t \neq 0$$

$$\text{and } t \mapsto \begin{cases} \sin(t)/t & \text{if } t \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

is continuous.

Hence, by composition, the limit of $f(x, y)$ when

$$(x, y) \rightarrow 0 \quad \text{is } 1.$$

Conclusion:

That function f is
continuous at $(0, 0)$.

Remark: And it is obviously continuous everywhere else. \square

\square

Exercise 3

- For $(x, y) \in \mathbb{R}^2 \setminus \{x = y\}$ (i.e. $(x, y) \in \mathbb{R}^2, x \neq y$)

$$f(x, y) = \exp\left(-\frac{1}{|x-y|}\right)$$

- By composition, f is clearly continuous on the two half-space

$$E_1 = \{(x, y) \in \mathbb{R}^2, x < y\}$$

$$\text{and } E_2 = \{(x, y) \in \mathbb{R}^2, x > y\}$$

- For any $y \in \mathbb{R}$, and $x \in \mathbb{R} \setminus \{y\}$ ($x \neq y$)

when $x \rightarrow y$, we have (for y being fixed!)

$$x - y \rightarrow 0$$

$$\text{so } |x - y| \xrightarrow{x \rightarrow y} 0^+$$

$$\text{so } \frac{1}{|x-y|} \xrightarrow{x \rightarrow y} +\infty \quad \text{so } \frac{-1}{|x-y|} \xrightarrow{x \rightarrow y} -\infty$$

Hence by composition (exp being continuous everywhere):

$$\text{we have } \exp\left(-\frac{1}{|x-y|}\right) \rightarrow 0^+$$

$$\text{So: } \forall y \in \mathbb{R}, \quad \lim_{\substack{x \rightarrow y \\ x \neq y}} f(x, y) = 0^+$$

Exercise 3 (concluding)

Hence, the value of $f(x, y)$ for $x = y$ have to be 0 for f to be continuous on \mathbb{R}^2 .

Exercise 6 (Partial)

a) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto |x| + |y|$.

+ f is continuous at $(0, 0)$: $|x| \rightarrow 0$ if $(x, y) \rightarrow (0, 0)$
 $|y| \rightarrow 0$

$$\text{so } f(x, y) \rightarrow f(0, 0) = 0$$

$$(x, y) \rightarrow (0, 0)$$

+ Does $\frac{\partial f}{\partial x}(0, 0)$ exist? For $h \neq 0$,

$$\frac{f(0+h, 0) - f(0, 0)}{h} = \frac{|h| + |0| - |0| + |0|}{h} = \frac{|h|}{h}$$

which does not have a limit when $h \rightarrow 0$.

So $\frac{\partial f}{\partial x}(0, 0)$ does not exist.

Conclusion: f is therefore not differentiable at $(0, 0)$.

Remark: Proving differentiability is harder!