

Mahindra École Centrale

MA 102: Linear Algebra and Applied Analysis

1 Linear Spaces

Defn 1.1. A nonempty set V along with two operations $+$ and \cdot is a *linear space* if V satisfies certain axioms (see pp. 3,4 of Apostol). The key axioms are *i*) If $x, y \in V$ then $x + y \in V$, *ii*) If $x \in V$ then $c \cdot x \in V$ for all $c \in \mathbb{F}$, and *iii*) There exists $0_V \in V$ such that $x + 0_V = x$ for all $x \in V$. A set $W \subseteq V$ is a *subspace* if W is a vector space with operations $+$ and \cdot .

Defn 1.2. For a set $S \subseteq V$, the span of S , $\text{span}(S)$, is the set of all finite linear combinations of elements of S . A finite set S is *dependent* if there exist $c_i, i = 1, \dots, n$ (not all zero), such that $\sum_{i=1}^n c_i x_i = 0$. S is independent if $\sum_{i=1}^n c_i x_i = 0$ implies that $c_i = 0, i = 1, \dots, n$.

Defn 1.3. A finite set S is a (*finite*) *basis* if S is independent and $\text{span}(S) = V$. The *dimension*, $\dim(V)$, of V is the number of elements in S .

Thrm 1.4. (i) Let V be a n -dimensional linear space. Then every finite basis for V has n elements.

(ii) Every set of independent elements in V can be expanded to a basis for V .

(iii) Every independent set of n elements is a basis.

Defn 1.5. $\langle \cdot, \cdot \rangle$ is an inner product if *i*) $\langle x, y \rangle = \langle y, x \rangle$, *ii*) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$, *iii*) $c \langle x, y \rangle = \langle cx, y \rangle$, and *iv*) $\langle x, x \rangle > 0, x \neq 0$. $\| \cdot \|$ is a Euclidean norm given by $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$. Finally, $x \perp y$ if $\langle x, y \rangle = 0$.

Thrm 1.6. *i*) $\|x\| > 0$ iff $x \neq 0$. *ii*) $\|cx\| = |c|\|x\|$. *iii*) $\|x + y\| \leq \|x\| + \|y\|$.

Defn 1.7. A set S is *orthogonal* if $x \perp y$ for all $x, y \in S, x \neq y$. An orthogonal set S is *orthonormal* if $\|x\| = 1$ for all $x \in S$. A set S is a *orthonormal basis* for V if S is a basis for V and it is orthonormal. For a set S , the orthogonal complement of S : $S^\perp = \{x \in V : x \perp y = 0, \text{ for all } y \in S\}$.

Thrm 1.8. Let V be a Euclidean space and let S be a finite-dimensional subspace of V . Then every element $x \in V$ is a sum of two unique elements $s \in S$ and $s^\perp \in S^\perp$. Furthermore, $s = \sum_{i=1}^n \langle x, e_i \rangle e_i$ is the best approximation of x on S with respect to $\langle \cdot, \cdot \rangle$ where $\{e_1, \dots, e_n\}$ is an orthonormal basis for S .

2 Linear Transformations

Defn 2.1. A map $T : V \rightarrow W$ is a *linear map* if *i*) $T(x + y) = T(x) + T(y), x, y \in V$ and *ii*) $T(cx) = cT(x), x \in V, c \in \mathbb{F}$. The range space $\mathcal{R}(T) \subseteq W$ is given by $\{y \in W, y = Tx, x \in V\}$. The null space

$\mathcal{N}(T) \subseteq V$ is given by $\{x : Tx = 0\}$. For a given $T \in \mathcal{L}(V, W)$, a *matrix* M of the linear map T is given by $M_{ji} = \langle f_j, T(e_i) \rangle$ where $\{e_i\}$ and $\{f_j\}$ are orthonormal bases of V and W , respectively.

Defn 2.2. Let $A \in \mathbb{F}^{n \times m}$. Then $\text{nullity}(A) = \dim(\mathcal{N}(A))$ and $\text{rank}(A) = \dim(\mathcal{R}(A))$.

Thrm 2.3. Let $T : V \rightarrow W$, let $\{e_i\}$ and $\{f_j\}$ be orthonormal bases of V and W , respectively, and M be the corresponding matrix of the linear map. Let $x \in V$ be such that $x = \sum c_i e_i$ then $T(x) = \sum d_j f_j$ where $d = Mc$ with $d = [d_1, \dots]^T$ and $c = [c_1, \dots]^T$.

Thrm 2.4. (The Rank-Nullity Theorem) If V is finite-dimensional then $\dim(\mathcal{N}(T)) + \dim(\mathcal{R}(T)) = \dim(V)$. Furthermore, $\text{nullity}(M) + \text{rank}(M) = \dim(V)$ where M denotes a matrix map of T .

Defn 2.5. Let $A \in \mathbb{F}^{n \times m}$. $B \in \mathbb{F}^{m \times n}$ is a *left inverse* of A if $BA = I_n$. If $m = n$, B is an *inverse* of A if $BA = AB = I_n$.

Defn 2.6. A matrix $A \in \mathbb{F}^{n \times n}$ is *nonsingular* (*singular*) if A is invertible (not invertible). A matrix $A \in \mathbb{R}^{n \times n}$ is *symmetric* (*skew-symmetric*) if $A = A^T$ ($A = -A^T$). For a matrix $A \in \mathbb{C}^{m \times n}$, $A^* \in \mathbb{C}^{n \times m}$ denotes the *complex-conjugate transpose* of A given by \bar{A}^T where \bar{A} denotes the complex-conjugate matrix of A . A matrix $A \in \mathbb{C}^{n \times n}$ is *Hermitian* (*skew-Hermitian*) if $A = A^*$ ($A = -A^*$).

Thrm 2.7. Let $A \in \mathbb{F}^{m \times n}$ and $b \in \mathbb{F}^m$. The following statements are equivalent:

- $Ax = b$ has a unique solution,
- $\text{rank}(A) = \text{rank}([A \ b]) = n$,
- $\mathcal{N}(A) = \{0\}$.

Thrm 2.8. Every invertible matrix can be written as a product of elementary matrices.

Defn 2.9. A function $\det: \mathbb{F}^{n \times n} \rightarrow \mathbb{F}$ is *determinant* if *i*) it is linear in columns (or rows), *ii*) it is zero if two columns (or rows) are the same, and *iii*) $\det(I_n) = 1$.

Defn 2.10. Let $A \in \mathbb{F}^{n \times n}$. (*i, j*)th *minor* $M_{ij}(A)$ is the $(n - 1) \times (n - 1)$ matrix with *i*th row and *j*th column removed from A . The (*i, j*)th *cofactor* $K_{ij}(A)$ is $(-1)^{i+j} \det(M_{ij}(A))$. The *adjugate matrix* $\text{Adj}(A)$ is the $n \times n$ matrix with (*i, j*)th element $K_{ji}(A)$.

Thrm 2.11. Let $A \in \mathbb{F}^{n \times n}$. Then $\det(A) = \sum A_{ij} K_{ij}(A)$ for any row or column.

Thrm 2.12. Let $A \in \mathbb{F}^{n \times n}$ and $\chi_A(x) = \det(xI_n - A)$. Then $\chi_A(A) = 0_{n \times n}$.