

## MA 102: Linear Algebra and Applied Analysis

Mid Term Examination II, March 27, 2015  
Duration: 180 min, Maximum Marks: 100

### Problem 1. (10 Marks)

- (1.a) Consider the set  $S$  of all vectors in  $\mathbb{R}^2$  with usual addition but with the multiplication given by  $c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ cy \end{bmatrix}$  for all  $c, x, y \in \mathbb{R}$ . Determine whether the set  $S$  forms a linear space. If it does not then state and justify which of the linear space axioms fail to hold.
- (1.b) Let  $V = \mathbb{R}^2$  and  $W = \left\{ \begin{bmatrix} a \\ |b| \end{bmatrix}; a, b \in \mathbb{R} \right\}$ . Check if  $W$  is a subspace of  $V$ .
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### Problem 2. (5 Marks)

Let  $V$  be the linear space of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Determine whether the set  $S \subset V$  is linearly independent where  $S = \{1 + x, x + x^2, 1 + x^2\}$ . Compute  $\text{span}(S)$  and  $\dim(\text{span}(S))$ .

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### Problem 3. (5 Marks)

Let  $V$  be an Euclidean space and let  $x, y \in V$ . Show that  $\langle x, y \rangle = 0$  if and only if  $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ .

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### Problem 4. (10 Marks)

Consider the set of three vectors  $\{X, Y, Z\}$  in  $\mathbb{R}^4$  given by

$$X = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, Y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, Z = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}.$$

- Show that these three vectors are mutually orthogonal.
  - Find a vector  $W$  such that the set  $\{X, Y, Z, W\}$  is a set of mutually orthogonal vectors.
  - Convert the resulting set into a mutually orthonormal basis.
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### Problem 5. (10 Marks)

Let  $V$  be the linear space of continuous functions  $f : [0, 2\pi] \rightarrow \mathbb{R}$  with inner product  $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)dx$ . Let  $f(x) = x^2$ , find the element  $g$  nearest to  $f$  in the subspace spanned by the set  $\{1, \cos(x)\}$ .

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### Problem 6. (10 Marks)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be defined by  $T(x, y, z) = (x + z, x + y + 2z, 2x + y + 3z)$ .

- Find null space and range space of  $T$  and their respective dimensions.
- Is  $T$  invertible?

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**Problem 7. (10 Marks)**

Let  $V$  be the linear space of polynomials of degree  $\leq 2$ . For  $p \in V$ ,  $T : V \rightarrow V$  is defined as  $T(p)(x) = p(x - 1)$  for all  $x \in \mathbb{R}$ . Is  $T$  linear? If  $T$  is linear then derive the matrix of this transformation with respect to standard ordered basis.

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**Problem 8. (5 Marks)**

Compute inverse of the matrix  $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$  using Gaussian elimination method.

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**Problem 9. (5 Marks)**

Let  $T \in \mathcal{L}(V, W)$ . Prove that  $\mathcal{N}(T)$  and  $\mathcal{R}(T)$  are subspaces of  $V$  and  $W$ , respectively.

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**Problem 10. (30 Marks)** Let  $V$  denote the set of all real sequences. That is, if  $v \in V$  then  $v_1, v_2, v_3, \dots$  forms a real number sequence.

(1.a) Show that  $V$  is a vector space by proving the key properties listed in the summary sheet. Is  $V$  finite or infinite-dimensional? Justify your answer.

(1.b) Let  $W \subseteq V$  be such that if  $w \in W$  then

$$w_{n+2} = w_{n+1} + w_n, \quad \text{for all } n = 0, 1, \dots \quad (\text{Eq.1})$$

Show that  $W$  is a subspace of  $V$ .

(1.c) Show that a sequence  $x_n = \lambda^n$ ,  $n = 0, 1, 2, \dots$  is an element of  $W$  if and only if  $\lambda^2 - \lambda - 1 = 0$ .

(1.d) Determine the two roots  $\lambda_1, \lambda_2 \in \mathbb{R}$  of the polynomial  $\lambda^2 - \lambda - 1$  and show that they are real and distinct. Next, let  $w$  be such that  $w_n = c_1 x_n + c_2 y_n$  where  $c_1, c_2 \in \mathbb{R}$ ,  $x_n = \lambda_1^n$ , and  $y_n = \lambda_2^n$ . Show that  $w \in W$  and express it in terms of  $c_1, c_2, \lambda_1$ , and  $\lambda_2$  by substituting the numerical values of  $\lambda_1$  and  $\lambda_2$ .

(1.e) Let  $w \in W$ . Prove the following two equations

$$\begin{aligned} w_n &= \lambda_1^n \left( \frac{\lambda_2 w_0 - w_1}{\lambda_2 - \lambda_1} \right) - \lambda_2^n \left( \frac{\lambda_1 w_0 - w_1}{\lambda_2 - \lambda_1} \right) \\ &= w_0 \left( \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1 - \lambda_2} \right) + w_1 \left( \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \right). \end{aligned} \quad (\text{Eq.2})$$

This can be shown by first, showing that (Eq.2) is trivially true for  $n = 0, 1$  and then show that (Eq.1) holds if  $w$  satisfies (Eq.2). (Do not forget that the values of  $\lambda_1$  and  $\lambda_2$  are at your disposal).

(1.f) Using the previous part, determine a basis for  $W$  and compute the dimension of  $W$ .

(1.g) Let  $F \in W$  be such that  $F_0 = F_1 = 1$ . Using (Eq.1), compute  $F_2, \dots, F_7$ . Identify this sequence. Show that

$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right], n = 0, 1, \dots$$

(1.h) Let  $w \in W$ . Show that

$$\begin{bmatrix} w_n \\ w_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} w_{n-1} \\ w_n \end{bmatrix}, \quad n = 1, 2, \dots, \quad (\text{Eq.3})$$

$$\begin{bmatrix} w_n \\ w_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, \quad n = 0, 1, \dots \quad (\text{Eq.4})$$

(1.i) Let  $a = \frac{1+\sqrt{5}}{2}$ ,  $b = \frac{1-\sqrt{5}}{2}$ ,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} b & 1 \\ -1 & b \end{bmatrix}, \quad D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Show that

(1)  $ab = -1$ ,

(3)  $S^{-1} = \frac{1}{a-b} \begin{bmatrix} -1 & -a \\ a & -1 \end{bmatrix}$ ,

(2)  $a + b = 1$ ,

(4)  $A = SDS^{-1}$ .

Finally show that

$$A^n = SD^nS^{-1} = \frac{1}{a-b} \begin{bmatrix} a^{n-1} - b^{n-1} & a^n - b^n \\ a^n - b^n & a^{n+1} - b^{n+1} \end{bmatrix}. \quad (\text{Eq.5})$$

(1.j) Substitute (Eq.5) into (Eq.4) and show that (Eq.4) and (Eq.2) are identical.