

# 1 First problem : about a family of 2-dimensional integrals

Let  $f$  be the function defined on  $\mathbb{R}_+ = [0, +\infty)$  by:

$$\forall r > 0, f(r) = \int_0^r e^{-u^2} du,$$

and  $I$  be the function defined on  $\mathbb{R}_+$  by:

$$\forall r > 0, I(r) = \int \int_{[0,r] \times [0,r]} e^{-x^2-y^2} dx dy.$$

We also define the domain  $D_r = \{(x, y) \in (\mathbb{R}_+) \times (\mathbb{R}_+) : 0 \leq x^2 + y^2 \leq r^2\}$ .

1. On a *drawing* on your answer sheet, draw some domains  $D_r$ , for the three values  $r = 1$ ,  $r = 2$  and  $r = 4$ .
2. By using Fubini's theorem, as seen in Prof. Satya's lectures, prove that, for any  $r > 0$ ,

$$I(r) = (f(r))^2.$$

3. Prove quickly the following set inclusions (for any value  $r > 0$ ):

$$D(r) \subset ([0, r] \times [0, r]) \subset D(\sqrt{2}r) \tag{1.1}$$

4. On a *new* drawing, illustrate the previous result (for an *abstract* value  $r > 0$ , not specified numerically on your drawing).
5. (*Nothing to be proven here, just read and use that theorem after*)

We admit the following result.

- Let  $\mathcal{A}$  be a 2-dimensional domain *included* in a bigger 2D domain  $\mathcal{B}$ :  $\mathcal{A} \subset \mathcal{B}$ .
- And consider  $g : \mathcal{B} \rightarrow \mathbb{R}$ , a function which takes only non-negative values (*i.e.*  $g(x, y) \geq 0$  for all  $(x, y)$  in its domain  $\mathcal{B}$ ).
- Then the integral of  $g$  on the *smaller* domain  $\mathcal{A}$  is *smaller* than the integral of  $g$  on the bigger domain  $\mathcal{B}$ :

$$\int_{\mathcal{A}} g(x, y) dx dy \leq \int_{\mathcal{B}} g(x, y) dx dy.$$

6. By using the given theorem (twice) and the last question (1.1), verify quickly but properly that the two following inequalities hold:

$$\int \int_{D(r)} e^{-x^2-y^2} dx dy \leq I(r) \leq \int \int_{D(\sqrt{2}r)} e^{-x^2-y^2} dx dy. \tag{1.2}$$

7. By using a change of variable (from Cartesian to polar coordinates), evaluate the two bounds of the previous inequality.

You should get a value for  $\int \int_{D(r)} e^{-x^2-y^2} dx dy$  and a value for  $\int \int_{D(\sqrt{2}r)} e^{-x^2-y^2} dx dy$ , both depending on the radius  $r$ .

8. Thanks to the squeezing theorem for real-value functions (of the variable  $r$ ), the values you got on the last question (7) and the inequalities 1.2, find the limit for  $I(r)$  when  $r \rightarrow +\infty$ .

## 2 About multi-variable functions

### 2.1 Continuity

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the function given by:

$$f(x, y) = \begin{cases} (x - y) \cos\left(\frac{1}{x + y}\right) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

**That function is still not correctly defined. What should I change?**

Show that  $f$  is **continuous** at the point  $(0, 0)$ .

### 2.2 Non-continuity

Let  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the function given by:

$$g(x, y) = \begin{cases} \frac{x + y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Show that  $g$  is **not continuous** at the point  $(0, 0)$ .

### 2.3 A more complicated example of continuity

Let  $L$  be the domain  $\{(x, y) \in \mathbb{R}^2 : x = y\}$  (i.e. the line  $y = x$ ).

Then let  $U$  be the domain  $U = \mathbb{R}^2 \setminus L = \{(x, y) \in \mathbb{R}^2 : x \neq y\}$ .

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the function given by:

$$f(x, y) = \begin{cases} \frac{\sin(x) - \sin(y)}{x - y} & \text{if } (x, y) \in D \text{ (ie if } x \neq y) \\ \cos(x) & \text{if } (x, y) \in U \text{ (ie if } x = y) \end{cases}$$

By using the **théorème des accroissements finis**, show that  $f$  is **not continuous** at the point  $(0, 0)$ .

### 2.4 TODO: one more problem

**We need one more question one directional derivative, partial derivative, and differentiability.**

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## 3 A simple 2-dimensional integral

Compute that integral carefully:

$$\iint_{\mathcal{A}} x^2 dx dy.$$

Where the region  $\mathcal{A}$  is given as  $\mathcal{A} = \{(x, y) \in \mathbb{R}^2 : x \geq 0, 1 \leq x^2 + y^2 \leq 2\}$ .

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## 4 A second 2-dimensional integral

Compute that other integral (and do it carefully):

$$\iint_{\mathcal{B}} x \cos(x) dx dy.$$

Where the region  $\mathcal{B}$  is the triangle given by the three points  $(0, 0)$ ,  $(\pi, 0)$  and  $(\pi, \pi)$ .

Is it enough to be unambiguous?

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## 5 Computing a 2D area

Compute the area of the domain  $D$  given by  $D = \{(x, y) \in \mathbb{R}^2 : y \leq x \leq y^2, 1 \leq y \leq 2\}$ .

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## 6 Computing a volume

Compute the volume defined by these equations and inequalities:

$$\begin{cases} z \geq 0 \\ x^2 + y^2 = x \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

Is it enough to be unambiguous?