

MA 102: Linear Algebra and Multi-variable Calculus

Mahindra École Centrale

Duration: 3 hours | *Final Semester Examination* | Total 100 marks

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Problem IV: Lagrange polynomial interpolation (Marks: 20)

This last problem is focusing on the Lagrange polynomials for an interpolation problem. Let f be a function of the real variable: $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto f(x)$. We are looking for *polynomials* (of degree $\leq n$) taking the same values as this function f at $(n + 1)$ points x_0, \dots, x_n (all distinct).

The problem is long, but the questions can all be answered quickly (but with rigor and justification, of course). The underlined parts help you to identify what you should do.

Problem IV: Part 1

Let $n \in \mathbb{N}, x_0, \dots, x_n \in \mathbb{C}$ all distinct. n and the x_i are *constants* in all this part.

Q.IV.1.1) We are looking for $(n + 1)$ polynomials L_0, \dots, L_n , all of degree $\leq n$, satisfying:

$$\forall 0 \leq i \leq n, \quad L_i(x_i) = 1. \tag{1}$$

$$\forall 0 \leq i, j \leq n, \quad i \neq j \implies L_i(x_j) = 0. \tag{2}$$

In this question, let i be one integer such that $0 \leq i \leq n$.

Q.IV.1.1.a) Thanks to the first condition (1), show that there exists a unique constant $C_i \in \mathbb{C}$ such that:

$$L_i(x) = C_i \times \prod_{0 \leq j \leq n, j \neq i} (x - x_j).$$

Q.IV.1.1.b) Thanks to the second condition (2) ($L_i(x_i) = 1$), give the value of this constant $C_i \in \mathbb{C}$.

Note: if you are unable to get this value, you can use the symbol C_i in the rest of the problem.

Q.IV.1.2) So in fact, Q.IV.1.1) just proved that if L_0, \dots, L_n are $(n + 1)$ polynomials of degrees $\leq n$, satisfying (1) and (2), then necessarily there exists $(n + 1)$ constants C_0, \dots, C_n such that:

$$\forall 0 \leq i \leq n, \quad \forall x \in \mathbb{R}, \quad L_i(x) = C_i \times \prod_{0 \leq j \leq n, j \neq i} (x - x_j).$$

Conversely, justify quickly that, with the values of C_i found in Q.IV.1.1.b), this family L_0, \dots, L_n is satisfying the two conditions.

So we have proved the *existence* and *uniqueness* of a family L_0, \dots, L_n satisfying both (1) and (2).

Q.IV.1.3) Let V be the (complex) vector space of the (complex) polynomials, of degree $\leq n$.
(*Note:* we do **not** ask you to justify that V is a vector space.)

Q.IV.1.3.a) Without justifying, give the dimension¹ of this vector space V .

Q.IV.1.3.b) Similarly, without justification, give a basis for the vector space V .

¹ Hint: think carefully about your answer. If you are unsure, check with some example values, like $n = 0$ or $n = 1$.

Q.IV.1.4) Let us prove existence and uniqueness of the Lagrange polynomials:

Q.IV.1.4.a) By using the conditions (1) and (2), show that² the family L_0, \dots, L_n (seen as *vectors* of the linear space V) is linearly independent.

Q.IV.1.4.b) Justify quickly but carefully that the family L_0, \dots, L_n is a basis of V .

This unique family of polynomials L_0, \dots, L_n are called the Lagrange polynomials of order n , for the interpolation problem at the points x_0, \dots, x_n .

Problem IV: Part 2

Q.IV.1.1) Let $n \in \mathbb{N}$, $x_0, \dots, x_n \in \mathbb{C}$ all distinct, and $y_0, \dots, y_n \in \mathbb{C}$ be $(n+1)$ complex values. n and the x_i are *constants* in all this part.

We are looking for a polynomial $P \in V$ that is satisfying $P(x_i) = y_i, \forall 0 \leq i \leq n$.

Q.IV.2.1.a) Let $t \in \mathbb{R}$ be any real number. By using what we already know about these polynomials L_0, \dots, L_n , prove that

$$P(t) = \sum_{i=0}^n P(x_i)L_i(t).$$

Hint: You could evaluate both sides on certain points of your choice, and use the fact that both sides are polynomial expressions (think about their degree!).

Q.IV.2.1.b) Thanks to the previous result, quickly conclude that we now have the following result:
 $\forall n \in \mathbb{N}, \forall x_0, \dots, x_n \in \mathbb{C}$ (all distinct), $\forall y_0, \dots, y_n \in \mathbb{C}$, there exists one and only (complex) polynomial P of degree $\leq n$, and which satisfies $\forall 0 \leq i \leq n, P(x_i) = y_i$.

Give its expression, ie. for $t \in \mathbb{R}$, what is $P(t)$?

Q.IV.1.2) Now let $f: \mathbb{R} \rightarrow \mathbb{R}, t \mapsto f(t)$, $n \in \mathbb{N}$, $x_0, \dots, x_n \in \mathbb{R}$ all distinct.

Define a good family of values $y_0, \dots, y_n \in \mathbb{R}$, in order to use the previous result (Q.IV.2.1.b) and prove that there exists a *unique* polynomial P of degree $\leq n$ which satisfies $P(x_i) = f(x_i) \forall 0 \leq i \leq n$.

Let us write $L_{f,n}$ for this polynomial P . It is called the interpolation polynomial of f at the order n , at the points x_0, \dots, x_n .

Q.IV.1.3) Here, we are considering some examples of Lagrange polynomial, for small values of $n \in \mathbb{N}$:

- Give the expression of $L_{f,0}(t)$ and its degree.
- Give the expression of $L_{f,1}(t)$ (and its degree). What is the condition for its degree to be exactly 1?
- Give an example of $f: \mathbb{R} \rightarrow \mathbb{R}, t \mapsto f(t)$, and x_0, x_1 , such that $L_{f,1}(t)$ is **not** of degree 1.
- Give the expression of $L_{f,2}(t)$ (and its degree).

Similarly, find a simple example of points x_0, x_1, x_2 and f such that this polynomial $L_{f,2}(t)$ is **not** of degree 2.

Q.IV.1.4) Writing the monomial terms (t^k) in the Lagrange basis:

Q.IV.1.5.a) Compute the sum $\sum_{i=0}^n L_i(t)$ for any real number $t \in \mathbb{R}$.

Q.IV.1.5.b) Similarly, show that:

$$\forall t \in \mathbb{R}, \forall 0 \leq k \leq n, t^k = \sum_{i=0}^n x_i^k L_i(t).$$

Hint: try to apply the result of the question Q.IV.2.1.b) to the polynomial $P(t) = t^k$.

Q.IV.1.5.c) These equations show that, for all $0 \leq k \leq n$, the vector t^k (of the linear space V) can be written as the vector $[x_i^k]_{0 \leq i \leq n} = [x_i^0, x_i^1, \dots, x_i^n] = [1, x_i, \dots, x_i^n]$ in the **basis** L_0, \dots, L_n .

Write the matrix of change of bases, from the basis $(L_i)_{0 \leq i \leq n}$ to the basis $(t^k)_{0 \leq k \leq n}$.

What is its size?

² *Hint:* Write a linear combination of the L_j , and try to evaluate it at the points x_i .

(End of the problem.)

- TODO: proof-read and conclude.
- TODO: remove some hints?