

MA 102: Linear Algebra and Applied Analysis

Final Exam, May 05, 2015
Duration: 180 min, Maximum Marks: 250

Problem 1. (5 marks) Using Stoke's theorem, evaluate $\oint_C F \cdot dr$ where $F(x, y, z) = (3x - y)\mathbf{i} - 2yz^2\mathbf{j} - 2y^2z\mathbf{k}$ over the surface of the sphere $x^2 + y^2 + z^2 = 16, z > 0$.

Answer:

Problem 2. (5 marks) Use Gauss divergence theorem to evaluate $\int \int_S F \cdot \hat{n} ds$ where $F(x, y, z) = 3x^2\mathbf{i} + 6y^2\mathbf{j} + z\mathbf{k}$.

Answer:

Problem 3. (5 marks) Determine the singular points of the following differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 6y = 0, x \in \mathbb{R}$$

and determine whether they are regular (weak) or irregular (strong).

Answer:

Problem 4. (5 marks) Transform the following first-order vector differential equation into a higher-order scalar equation:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Answer:

Problem 5. (5 marks) Consider the (vibrational) linear system given by $\ddot{x}(t) + 2\zeta\omega\dot{x}(t) + \omega^2x(t) = 0$ where $\omega > 0$. For what values of ζ is this system asymptotically stable?

Answer (with justification):

Problem 6. (5 marks) Let $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \mathbf{w}$ be the inner product defined on \mathbb{R}^2 .

(i) Find all vectors in \mathbb{R}^2 that are orthogonal to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(ii) Find an orthonormal basis with respect to this inner product.

Answer:

Problem 7. (5 marks) Define an inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mathbf{v}$ on \mathbb{R}^2 , where $a > 0$ and $b > 0$. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Find values of a and b such that \mathbf{u}_1 and \mathbf{u}_2 are orthogonal with respect to the above inner product.

Answer:

Problem 8. (5 marks) Prove that every eigenvalue of a nilpotent matrix is zero.

Answer:

Problem 9. (5 marks) For what numbers b and c are the following matrices positive definite?

(i) $\begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix}$.

Answer:

Problem 10. (5 marks) Let A be positive definite and Q be an orthogonal matrix. Determine whether the following statements are true or false.

(i) $Q^T A Q$ is a diagonal matrix.

(ii) $Q^T A Q$ is positive definite.

(iii) $Q^T A Q$ has the same eigenvalues as A .

(iv) e^{-A} is positive definite.

Answer:

Problem 11. (5 Marks) Find the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Answer:

Problem 12. (5 Marks) Why would the eigenvectors of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$$
 be linearly independent?

Answer (justify by computing the relevant numbers):

Problem 13. (5 Marks) Consider the Cauchy-Schwarz inequality relating the inner product of two vectors with their norms. In the case when the equality holds, what can you say about the two vectors in terms of linear independence?

Answer (provide a brief Justification):

Problem 14. (5 Marks) A mathematics student chooses to write the Schur decomposition of a certain matrix as follows: $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$. What is wrong with her/his claim that this is a Schur decomposition?

Answer:

Problem 15. (5 Marks) Let v_1, v_2, v_3 , and v_4 be vectors in \mathbb{R}^3 . Determine whether these vectors are linearly independent or dependent. Justify.

Answer:

Problem 16. (5 Marks) If $\det(A) = -3$, and A is 3×3 , compute $\det(A/2)$.

Answer:

Problem 17. (5 Marks) Give an example consisting of two orthogonal vectors in \mathbb{R}^2 which are not linearly independent.

Answer:

Problem 18. (5 Marks) Show that the eigenvalue of a square matrix belonging to a given eigenvector is unique (**Hint:** Only non-zero eigenvectors are admitted).

Answer:

Problem 19. (5 Marks) Let A be an invertible square matrix. Show that the eigenvalues of A^{-1} are reciprocals of the eigenvalues of A and that the corresponding eigenvectors are the same. (Give a short proof using the usual symbols. Do not write long sentences.)

Answer:

Problem 20. (5 Marks) Convert the following matrix into a positive definite matrix by adding a suitable matrix to it: $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ (Show your work. The solution is not unique.)

Answer:

Problem 21. (10 marks) Let \mathbb{R}^∞ denote the linear space of real sequences with usual addition and multiplication. Answer the following

- (i) Let U be the set of all arithmetic progressions. Show that U is a subspace of \mathbb{R}^∞ .
- (ii) Let V be the set of all geometric progressions. Is V a subspace of \mathbb{R}^∞ ? Justify.

Answer:

Problem 22. (10 marks) Argue why the following subsets of \mathbb{R}^2 are not subspaces?

- (i) The unit circle centered at origin and its interior.
- (ii) A line in \mathbb{R}^2 that does not go through origin.
- (iii) The first quadrant of \mathbb{R}^2 .

Answer:

Problem 23. (10 marks) Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Determine whether the following statements are true or false. Explain

- (i) If $n > m$, then T is not one-to-one.
- (ii) If $n \leq m$, then T is one-to-one.
- (iii) If T is one-to-one and onto, then $m = n$.

Answer:

Problem 24. (10 marks) Let V be the vector space of functions from \mathbb{R} to \mathbb{R} . Now, let $Q : V \rightarrow V$ be given by $Q(f) = g$ where $g(x) = f(x) - f(-x)$ for all $x \in \mathbb{R}$

- (i) Show that Q is a linear map.
- (ii) What is the null space of Q ?
- (iii) What is the range space of Q ?

Answer:

Problem 25. (10 marks) Let $\langle A, B \rangle = \text{tr}(A^T B)$ denote an inner product on $\mathbb{R}^{2 \times 2}$. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

- (i) Compute $\langle A, B \rangle$ and $\langle B, A \rangle$.

(ii) Compute $\|A - B\|$.

(iii) Find a vector orthogonal to A .

Answer:

Problem 26. (10 marks) Let $A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$. Compute eigenvalues and corresponding (independent, if possible) eigenvectors of A . Is A diagonalizable?

Answer:

Problem 27. (10 marks) Let $A = \begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}$. Evaluate A^k (in terms of k).

Answer:

Problem 28. (10 marks) Let $A \in \mathbb{R}^{n \times n}$ be an orthogonal matrix, that is, $A^T A = I$.

(i) What are the possible values for the eigenvalues of A ?

(ii) What is the rank of A ?

(iii) What are the possible values for $\det(A)$?

Answer:

Problem 29. (10 Marks) Show that a normal matrix can be expressed as $X\Lambda X^*$ where Λ is a diagonal matrix and $XX^* = I$, and X and I have their usual meanings. (Use self explanatory symbols and write as little as possible. Do not write long sentences.)

Answer:

Problem 30. (10 Marks) Let A be a 2×2 matrix. Show that A^n may be expressed as a linear combination of A and the identity I_2 , where n is an integer ≥ 2 . (Use self explanatory symbols and write as little as possible. Do not write long sentences.)

Answer:

Problem 31. (10 Marks) Write one eigenvalue of the following matrix and justify your an-

swer:
$$\begin{bmatrix} 0 & -1 & -2 & -3 & -4 \\ 1 & 0 & -1 & -2 & -3 \\ 2 & 1 & 0 & -1 & -2 \\ 3 & 2 & 1 & 0 & -1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

Answer (with justification):

Problem 32. (10 marks) Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by $F(x, y, z) = -xy \mathbf{i} + y^2 \mathbf{j} + z \mathbf{k}$. Using line integrals, determine the work done by the force F in moving a particle over the circular path $x^2 + y^2 = 4, z = 0$ from $(2, 0, 0)$ to $(0, 2, 0)$.

Answer (with justification):

Problem 33. (10 marks) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Compute e^{A+A^T} , e^A , and e^{A^T} . Check if $e^{A+A^T} = e^A \cdot e^{A^T}$. Justify

Answer:

Problem 34. (10 marks) Let A be a diagonalizable $\mathbb{C}^{2 \times 2}$ matrix. Determine a necessary and sufficient conditions (in terms of eigenvalues of A) such that $\lim_{n \rightarrow \infty} A^n = 0$. What if $A = \lambda I_2 + N$ where $\lambda \in \mathbb{C}$ and $N^2 = 0$. Finally, **guess** the condition for $A \in \mathbb{C}^{n \times n}$.

Answer (with justification):

Problem 35. (10 marks) Consider the third-order differential equation

$$\frac{d^3x}{dt^3} + a \frac{d^2x}{dt^2} + b \frac{dx}{dt} + x = 0.$$

Show that the characteristic polynomial of the ODE is same as the characteristic polynomial of the corresponding state-space matrix.

Answer (with justification):

THE END
