



**Mahindra  
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COLLEGE OF ENGINEERING

# MA 102: Linear Algebra and Analysis

Key Concepts and Results

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# Outline



Linear Algebra

Matrix Mathematics

System of Linear Differential Equations



## Linear Spaces

- ▶ Closure, Associativity, Commutativity, Additive Identity & Inverse etc
- ▶ The scalars can be real or complex numbers (in general, any field)

## Subspaces

- ▶ **Subspace** is a subset of a vector space which is a vector space itself
- ▶ **Fact:** Given a subset of a vector space, closure axioms are sufficient to check if a subset is a subspace

## Dependent and Independent Sets

- ▶ If  $\sum_{i=1}^n c_i v_i = 0$  (with **not all zero**  $c_i$ ) then the **set**  $\{v_1, \dots, v_2\}$  is **dependent**
- ▶ If  $\sum_{i=1}^n c_i v_i = 0$  implies  $c_i = 0$  then  $\{v_1, \dots, v_2\}$  is **independent**
- ▶ **Span of a set**  $S$  is the set of all **finite** linear combinations of elements of  $S$

## Basis and Dimension

- ▶ An independent **set**  $S$  is a **basis** for a linear space  $V$  if  $\text{span}(S) = V$
- ▶ The number of elements of basis is **dimension** of the linear space

- ▶ **Fact:** If  $n = \dim(V)$  then every set with  $m > n$  elements is dependent
- ▶ **Fact:** If  $n = \dim(V)$  then every independent set with  $n$  elements is basis
- ▶ **Fact:** If  $n = \dim(V)$  then every independent set with  $m < n$  elements can be extended into a basis

## Examples

- ▶  $\mathbb{R}^3$ : Unit vectors in  $x$ ,  $y$ , and  $z$ -axis form the canonical basis
- ▶ Set of polynomials:  $\{1, x, x^2, \dots\}$  forms a basis of this infinite-dimensional space

## Inner Product

- ▶  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$  is **inner product** on  $V$  if *i*)  $\langle x, y \rangle = \langle y, x \rangle$ ,  
*ii*)  $\langle cx, y \rangle = c\langle x, y \rangle$ , *iii*)  $\langle x, x \rangle > 0, x \neq 0$ ,  
*iv*)  $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

## Euclidean Norm

- ▶  $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$ : *i*)  $\|cx\| = c\|x\|$ , *ii*)  $\|x\| > 0$  for  $x \neq 0$ ,  
*iii*)  $\|x + y\| \leq \|x\| + \|y\|$

## Cauchy-Schwarz Inequality

- ▶  $|\langle x, y \rangle| \leq \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}$
- ▶  $|\langle x, y \rangle| \leq \|x\| \|y\|$
- ▶ Equality holds iff  $x, y$  are dependent

## Orthogonality and Orthonormal Basis

- ▶  $x$  is orthogonal to  $y$  ( $x \perp y$ ) if  $\langle x, y \rangle = 0$
- ▶ A set is orthogonal if all elements are mutually orthogonal
- ▶ An orthogonal set is orthonormal if every element has unit norm
- ▶ An orthonormal set that is a basis is an orthonormal basis

## Key Facts

- ▶ **Fact:** Every orthogonal set with nonzero elements is independent
- ▶ **Fact:** Gram-Schmidt algorithm generates orthogonal sets

## Definitions

- ▶  $T : V \rightarrow W$  is a **linear map** if  $T(x + y) = T(x) + T(y)$  and  $T(cx) = cT(x)$ . If  $V = W$  then  $T$  is a **linear operator**
- ▶  $\mathcal{N}(T) \subseteq V$  is the **null space** of  $T$  given by  $\{x \in V : T(x) = 0\}$
- ▶  $\mathcal{R}(T) \subseteq W$  is the **range space** of  $T$  given by  $\{y \in W : y = T(x), x \in V\}$
- ▶  $\dim(\mathcal{N}(T))$  is the **nullity** or **defect** of  $T$
- ▶  $\dim(\mathcal{R}(T))$  is the **rank** of  $T$





## Key Results

- ▶ **Fact:**  $\mathcal{N}(T)$  and  $\mathcal{R}(T)$  are subspaces of  $V$  and  $W$ , respectively
- ▶ **Rank-Nullity Theorem:**  $\dim(\mathcal{N}(T)) + \dim(\mathcal{R}(T)) = \dim(V)$
- ▶ **Fact:** If  $T$  is **one-to-one** (that is,  $\mathcal{N}(T) = \{0\}$ ) then  $\dim(W) \geq \dim(V)$
- ▶ **Fact:** If  $T$  is **onto** (that is,  $\mathcal{R}(T) = W$ ) then  $\dim(W) \leq \dim(V)$
- ▶ **Fact:** If  $T$  is **invertible** (one-to-one & onto) then  $\dim(W) = \dim(V)$

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## Existence and Uniqueness

- ▶ Let  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$
- ▶ **Fact:**  $Ax = b$  has a solution if and only if  $\text{rank}(A) = \text{rank}([A \ b])$
- ▶ **Fact:**  $Ax = b$  has a unique solution iff  $\mathcal{N}(A) = \{0\}$  iff  $\text{rank}(A) = \text{rank}([A \ b]) = n$
- ▶ **Fact:** If  $m = n$ ,  $Ax = b$  has a unique solution iff  $A$  is invertible

## Facts on Rank

- ▶ **Fact:**  $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$
- ▶ **Fact:**  $\text{rank}(AB) = \text{rank}(A)$  if  $B$  has full rank

## Solution Techniques and Related Concepts

- ▶ **Gaussian elimination:** elementary row operations, forward and backward substitutions
- ▶ **Fact:** Elementary row operations are equivalent to multiplication by elementary matrices
- ▶ **Fact:** Inverse of an elementary matrix is an elementary matrix
- ▶ **Fact:** Permutation matrices are idempotent ( $P^2 = I$ )
- ▶ **Fact:** Every invertible matrix can be written as a product of elementary matrices



- ▶ Diagonal, triangular, block-diagonal, block-triangular
- ▶ Symmetric:  $A = A^T$ , skew-symmetric:  $A = -A^T$
- ▶ Hermitian:  $A = A^*$ , skew-Hermitian:  $A = -A^*$
- ▶ Idempotent:  $A^2 = A$ , nilpotent:  $A^k = 0$  for some  $k \geq 1$
- ▶ Normal:  $A^T A = A A^T$  or  $A^* A = A A^*$
- ▶ Orthogonal (orthonormal?):  $A^T A = A A^T = I$
- ▶ Unitary:  $A^* A = A A^* = I$
- ▶ Positive (nonnegative) definite:  $A > 0$  ( $A \geq 0$ ) if  $A$  is Hermitian and  $x^* A x > 0$ ,  $x \neq 0$  ( $x^* A x \geq 0$ )

## Determinant

- ▶ **Fact:**  $\det(A) = 0$  iff  $A$  is **singular**
- ▶ **Fact:**  $\det(A) \neq 0$  iff  $A$  is **invertible**
- ▶ **Fact:** Let  $A \in \mathbb{F}^{n \times m}$ ,  $B \in \mathbb{F}^{m \times n}$ , and  $\lambda \in \mathbb{F}$

$$\lambda^m \det(\lambda I_n + AB) = \lambda^n \det(\lambda I_m + BA)$$

## Eigenvalues, Eigenvectors, & Singular Values

- ▶ If  $Ax = \lambda x$  for  $x \neq 0$  then  $\lambda$  is **eigenvalue**,  $x$  is **eigenvector**
- ▶  $\sigma$  is a **singular value** of  $A$  if  $\sigma^2 \in \text{spec}(A^*A)$
- ▶ **Fact:**  $\lambda \in \text{spec}(A)$  iff  $\det(\lambda I - A) = 0$
- ▶ **Fact:** If  $\lambda \in \text{spec}(A)$  then  $\lambda^n \in \text{spec}(A^n)$  for all  $n \geq 1$

## Diagonalizability

- ▶  $A$  is **diagonalizable** if  $A = X\Lambda X^{-1}$  for  $\Lambda$  diagonal and nonsingular  $X$
- ▶ **Fact:** In this case, elements of  $\Lambda$  are eigenvalues, columns of  $X$  are corresponding eigenvectors
- ▶ **Fact:**  $A \in \mathbb{F}^{n \times n}$  is diagonalizable iff  $A$  has  $n$  independent eigenvectors
- ▶ **Fact:** If  $A$  has  $n$  **distinct eigenvalues** then  $A$  is diagonalizable
- ▶ **Fact:** If  $A$  is **normal** then  $A$  is diagonalizable
- ▶ **Remark:**  $A$  need not have distinct eigenvalues for diagonalizability (consider  $I_n$ )

## Decompositions

- ▶ **Schur Decomposition:**  $A = SUS^*$ ,  $S$  is unitary and  $U$  is upper triangular
- ▶ **Singular Value Decomposition:**  $A = U\Sigma V^*$ , where  $U, V$  are unitary,  $\Sigma$  is diagonal with elements as singular values
- ▶ **Jordan Canonical Form:**  $A = SJS^*$ , where  $S$  is unitary,  $J = \text{block-diag}(J_1, \dots, J_m)$ ,  $J_i$  is a scalar block if  $\lambda_i$  is not repeated. Otherwise,  $J_i = \lambda_i I + N_i$ , where  $N_i$  is nilpotent
- ▶ **Polar Decomposition:**  $A = MU$  where  $M \geq 0$  and  $U$  is unitary
- ▶ **Euler's Identity:**  $A = Me^{j\Theta}$  where  $M \geq 0$  and  $\Theta$  is Hermitian





## The Cayley-Hamilton Theorem

- ▶ Let the **characteristic polynomial** be given by  
 $\chi_A(\lambda) = \det(\lambda I - A)$
- ▶ Then  $\chi_A(A) = 0$
- ▶ **Corollary:** Given  $A \in \mathbb{F}^{n \times n}$  and  $k \geq 0$ ,

$$A^{n+k} = c_0 I + c_1 A + c_2 A^2 + \cdots + c_{n-1} A^{n-1}$$

- ▶ **Fact:** If  $p(\cdot)$  is a polynomial then  
 $p(A) = Xp(\Lambda)X^{-1} = Sp(U)S^* = Sp(J)S^*$

## Key Results

- ▶ **Fact:** Eigenvalues of symmetric (Hermitian) matrices are real
- ▶ **Fact:** Eigenvalues of skew-symmetric (skew-Hermitian) matrices are imaginary
- ▶ **Fact:** Eigenvalues of orthogonal or unitary matrices has unit magnitude
- ▶ **Fact:** Eigenvalues of positive definite matrices are positive
- ▶ **Fact:**  $A \in \mathbb{R}^{2 \times 2}$  is positive definite iff  $A$  is symmetric,  $\text{tr}(A) > 0$ , and  $\det(A) > 0$

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## Definition and Key Facts

- ▶ Let  $A \in \mathbb{F}^{n \times n}$ . Then

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2!} + \dots + \frac{A^n}{n!} + \dots$$

- ▶ **Fact:**  $e^{\lambda I} = e^\lambda I \implies e^{0_{n \times n}} = I_n$
- ▶ **Fact:**  $e^{(x+y)A} = e^{xA}e^{yA} \implies e^A e^{-A} = I$
- ▶ **Fact:**  $\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$
- ▶ **Fact:** If  $AB = BA$  then  $e^{A+B} = e^A e^B = e^B e^A$



## Key Methods

- ▶ **Fact:** If  $A = X\Lambda X^{-1}$  then  $e^A = Xe^\Lambda X^{-1}$
- ▶ **Fact:** If  $N^k = 0$  then  $e^{(\lambda I+N)t} = e^{\lambda t} \sum_{i=0}^{k-1} \frac{N^i t^i}{i!}$

$$\sum_{i=0}^{n-1} \frac{N^i t^i}{i!} = \begin{bmatrix} 0 & t & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 0 & \cdots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$



## The Homogeneous Case

- ▶ **Fact:**  $\dot{x}(t) = Ax(t)$ ,  $x(0) = x_0$  has a unique solution  
 $x(t) = e^{At}x_0$
- ▶ **Fact:**  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  iff  $\operatorname{Re}(\lambda) < 0 \forall \lambda \in \operatorname{spec}(A)$
- ▶ **Fact:** For  $A \in \mathbb{R}^{2 \times 2}$ ,  $\operatorname{Re}(\lambda) < 0$  iff  $\operatorname{tr}(A) < 0$  and  $\det(A) > 0$
- ▶ **Fact:** If  $A$  has only imaginary eigenvalues then  $x(t)$  will be harmonic

## The Nonhomogeneous Case

- **Fact:**  $\dot{x}(t) = Ax(t) + Bu(t)$ ,  $x(0) = x_0$  has a unique solution

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- **Fact:** If  $u(t) = \sin(\omega t)$  and  $i\omega \in \text{spec}(A)$  then resonance occurs

## Second-Order Differential Equation

- **Fact:**  $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$  can be transformed to

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -c \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

where  $x_1 = x$  and  $x_2 = \dot{x}$



## Higher-Order Differential Equation

- **Fact:**  $x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \dots + a_1\dot{x}(t) + a_0x(t) = 0$  can be transformed to

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix},$$

where  $x_1 = x$ ,  $x_2 = \dot{x}$ ,  $\dots$ ,  $x_n = x^{(n-1)}$