

MA 101 : 2014

Final Examination.

Problem 4.

(30 marks)

Hand-written (complete) solution
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Part A.

1) * For $p, k \in \mathbb{N}^*$, $p \geq 1$,

so $x \mapsto \frac{1}{x^p}$ is decreasing because

f is differentiable, and $f'(x) = \frac{-p}{x^{p+1}} < 0$.

on $[1, +\infty)$ so on $[k, k+1]$

* Because f is decreasing on $[k, k+1]$.

$$\min_{[k, k+1]} f(x) = f(k+1) = \frac{1}{(k+1)^p}$$

$$\text{and } \max_{[k, k+1]} f(x) = f(k) = \frac{1}{k^p}$$

So the min-max inequality, applied to f .

on $[k, k+1]$, gives

$$\frac{1}{(k+1)^p} \leq \int_k^{k+1} \frac{1}{t^p} dt \leq \frac{1}{k^p} \quad (1)$$

Problem 4

Part A

2) For $p \geq 1$, $\frac{1}{x^p}$ is well defined for all $x > 1$,
so the finite sum $S_n(p) = \sum_{k=1}^n \frac{1}{k^p}$

is well defined.

We know that

$$[1, n] = \bigcup_{k=1}^{n-1} [k, k+1]$$

So by summing the inequality (1)

for $k=1 \dots n-1$,

$$\sum_{k=1}^{n-1} \left(\frac{1}{k+1}\right)^p \leq \int_1^n f(x) dx \leq \sum_{k=1}^{n-1} \left(\frac{1}{k}\right)^p$$

So

$$S_n(p) - 1 \leq \int_1^n f(x) dx \leq S_{n-1}(p)$$

3) For $A > 1$, A
 $g(A) = \int_1^A f(x) dx = \int_1^A \frac{1}{x^p} dx$

① If $p > 1$

$$= \frac{1}{p-1} \left[\frac{1}{x^{p-1}} \right]_1^A = \frac{A^{1-p}}{p-1} - \frac{1}{p-1}$$

So for $A \rightarrow +\infty$,
 $1-p < 1$

so $\frac{A^{1-p}}{p-1} \xrightarrow{A \rightarrow +\infty} 0$

so $g(A) \rightarrow \frac{1}{1-p}$

$$g(A) \rightarrow \frac{1}{1-p}$$

Problem 4

Part A

② If $p = 1$: $g(A) = \int_1^A \frac{1}{x} dx$

$$= \ln(A) \quad \begin{matrix} \longrightarrow +\infty \\ A \longrightarrow +\infty \end{matrix}$$

Conclusion $g(A)$ has a finite limit for $A \rightarrow +\infty$ if and only if $p > 2$.

4) If $p > 2$ $g(n)$ has a finite limit if when $n \rightarrow +\infty$,

so (2) gives:

$$g(n) \leq S_{n-1}(p) = S_{n-1}(p) - 1 + 1 \leq g(n-1) + 1$$

But now $(g(n-1))$ is bounded because convergent
And $(S_n(p))_{n \in \mathbb{N}^*}$ is increasing and bounded

Conclusion $(S_n(p))_n$ converges.

If $p < 2$ ($\text{à } p=1$) (2) gives

$$g(n) \leq S_{n-1}(p)$$

$$\downarrow_{\substack{n \rightarrow +\infty \\ +\infty}}$$

$$\text{So } S_{n-1}(p) \xrightarrow[n \rightarrow +\infty]{+\infty}$$

Problem 4

Part A

4)

Conclusion. The sequence $S(p) = (S_n(p))_p$ is
 convergent if $p \geq 2$,
 divergent (to $+\infty$) if $p < 2$,

so convergent if and only if $p \geq 2$.

Part B

1)
$$\varphi_0(t) = \begin{cases} -1 & \text{if } t = 0 \\ \frac{t^2/2\pi - t}{2\sin(t/2)} & \text{if } t \in (0, \pi] \end{cases}$$

Continuity: If $t \in (0, \pi]$, $\varphi_0(t)$ is continuous,
 Only $t=0$ is a potential problem.

For $t \rightarrow 0^+$,
$$\frac{t^2/2\pi - t}{2\sin(t/2)} = \frac{t/2}{\sin(t/2)} \times \left(\frac{t}{2\pi} - 1\right)$$

and
$$\frac{t}{2\pi} - 1 \rightarrow -1$$

$$\left\{ \begin{array}{l} \frac{t/2}{\sin(t/2)} \xrightarrow{t \rightarrow 0^+} 1 \end{array} \right.$$

so
$$\varphi_0(t) \xrightarrow{t \rightarrow 0^+} -1 = \varphi_0(0)$$

So φ_0 is indeed continuous on $[0, \pi]$.

Differentiability: For $t \in (0, \pi]$, φ_0 is differentiable at t ,

and
$$\varphi_0'(t) = \frac{\left(\frac{t}{\pi} - 1\right)}{2\sin(t/2)} - \frac{\left(\frac{t^2}{2\pi} - t\right) \cos(t/2)}{4\sin^2(t/2)}$$

$$\rightarrow \frac{1}{2\pi} \text{ if } t \rightarrow 0^+ \quad \text{so } \underline{\varphi_0'(0) \text{ exists.}}$$