

MA 101 (2014)
Final Examination

Problem 3

(30 marks)

Hand-written solution
by Prof. Lillian BESSON.
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Part A

1) $\mathcal{D}_p = \mathbb{R} \setminus \{0\} = \mathcal{D}_Q = (-\infty, 0) \cup (0, +\infty)$
with $P(x) = \frac{2}{x}$, $Q(x) = \frac{\cos(x)}{x^2}$.

But $\pi \in (0, +\infty)$ so the largest interval where the existence and uniqueness theorem is applicable is $(0, +\infty)$.

2) Directly thanks to the method seen in class,

$y(x) = C * \exp(-\int P(t) dt)$ for any $C \in \mathbb{R}$
are the solutions of $y' + P(x)y = 0$

But $-\int P(t) dt = -\int \frac{2}{t} dt = -2 \ln(t) + C$

so $y(x) = C * \exp(-2 \ln(x)) = \frac{C}{x^2}$

Conclusion: The solutions for the homogeneous equation are $y(x) = \frac{C}{x^2}$, defined on $(0, +\infty)$, for any $C \in \mathbb{R}$ constant.

Problem-3Part A.

3) We write $y(x)$ as $\frac{\varphi(x)}{x^2}$.

y is solution of the non-homogeneous equation
 $y' + \frac{2}{x}y = \frac{\cos(x)}{x^2} \Leftrightarrow \frac{\varphi'(x)}{x^2} = \frac{\cos(x)}{x^2} \quad \forall x > 0$

$$\Leftrightarrow \varphi'(x) = \cos(x) \quad \forall x > 0$$

$$\Leftrightarrow \varphi(x) = \sin(x) + C \quad \text{for any constant } C.$$

Conclusion: The solutions for the non-homogeneous equation $y' + \frac{2}{x}y = \frac{\cos x}{x^2}$ are

$$y(x) = \frac{\sin(x) + \varphi}{x^2} \quad \text{for any } \varphi \in \mathbb{R}$$

(defined on $(0, +\infty)$).

$$\text{So } y(\pi) = 0 \Leftrightarrow \frac{\sin(\pi) + \varphi}{\pi^2} = \frac{\varphi}{\pi^2} = 0$$

$$\Leftrightarrow \varphi = 0$$

Conclusion: The unique solution to the given IVP is $y(x) = \frac{\sin(x)}{x^2}$ defined on $(0, +\infty)$

4) For any $\varphi \in \mathbb{R}$, if $y(x) = \frac{\sin(x) + \varphi}{x^2}$

$$y'(x) + \frac{2}{x}y(x) = \frac{\cos(x)}{x^2} + \frac{-2\varphi}{x^3} + \frac{2\varphi}{x^3} = \frac{\cos(x)}{x^2} \quad \checkmark$$

$$\text{And if } \varphi = 0, \quad y(\pi) = \frac{\sin(\pi)}{\pi^2} = 0 \quad \checkmark$$

Conclusion: The given solutions are indeed solutions.

Problem 3

Part B.

1) For the homogeneous equation, we first solve the characteristic equation:

$$r^2 - 3r + 2 = 0 \quad (\Leftrightarrow) \quad r = 1 \text{ or } 2$$

So we can directly conclude that the general solutions are

$$y(x) = C_1 \cdot e^x + C_2 \cdot e^{2x} \quad \text{on } \mathbb{R}$$

For any two constants $C_1, C_2 \in \mathbb{R}$.

2) i) We use, as seen in class, the method of variation of the constants.

Let $y_1(x) = e^x$ $y_2(x) = e^{2x}$ - $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} (x)$

$$y(x) = C_1(x)y_1(x) + C_2(x)y_2(x)$$

is solution if and only if

$$= \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0$$

$$\begin{cases} C_1' y_1 + C_2' y_2 = 0 \\ C_1' y_1' + C_2' y_2' = f(x) = x^2 3x \end{cases}$$

$$\Leftrightarrow \begin{cases} C_1'(x) = -x^2 3 e^{-x} \\ C_2'(x) = x^2 3 e^{2x} \end{cases}$$

To be concluded.

by you! december break homework!!