

MA 101 (2014)

Final Examination.

Wednesday the 3rd of December

Problem 1. (30 marks)

quick hand-written solution
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1.1)

(B) -1

because $z: (A), e^{i\theta}$ for $\theta = \frac{\pi}{2}$, $1+i: (C)$
are complex numbers, which cannot be compared
to real numbers (with $<$).

1.2)

(B) No intrinsic meaning.

1.3)

(C) None of the above:

- * A is wrong, for example if $x=0, y=-1$
 $z = -i \quad \arg(z) = -\frac{\pi}{2} + 2k\pi \quad \forall k \in \mathbb{Z}$.
- * B is only correct if $x \neq 0$ and $y > 0$.
- * C is wrong because $\arg(z)$ is not defined if $z = 0$.

1.4)

(C)

By ratio test.

A) is wrong, example

with $a_n = \frac{1}{n}$

B) is A).

D) is obviously wrong: if $a_n \not\rightarrow 0, \sum_{n=0}^{\infty} a_n$ diverges

(1)

Problem 1

1.5) (A)

$$x_0 = 0, \quad x_{n+1} = s + x_n^2$$

$$x_1 = s, \quad x_2 = s + s^2, \quad x_3 = s + (s + s^2)^2 = s + s^2 + 2s^3 + s^4$$

If $(x_n)_{n \in \mathbb{N}}$ converges, to a limit $x_\infty \in \mathbb{R}$

we have $x_{n+1} \rightarrow x_\infty$ also,

so $x_\infty = s + (x_\infty)^2$ i.e. $(x_\infty)^2 - x_\infty + s = 0$

i.e. the quadratic equation $r^2 - r + s = 0$ has at least one real root. (i.e. $\Delta \geq 0$)

But $\Delta = (-1)^2 - 4 \times 1 \times s = 1 - 4s \geq 0$

$(\Rightarrow) s \leq 1/4$

1.6) We use L'Hôpital rule:

(D)

$$\frac{\ln(x)}{x-\beta} \underset{x \rightarrow 0^+}{\sim} \frac{1/x}{-\beta x^{-\beta-1}} = \frac{-1/\beta}{x-\beta}$$

but $-1/\beta > 0$
 $x^{-\beta} \rightarrow +\infty$
 $x \rightarrow 0^+$
 $\frac{1}{x-\beta} \rightarrow 0^+$
 $x \rightarrow 0^+$

so $\frac{1}{x-\beta} = x^\beta$

so $\frac{-1/\beta}{x-\beta} \rightarrow 0^-$
 $x \rightarrow 0^+$

i.e. $\frac{\ln(x)}{x-\beta} \rightarrow 0^-$
 $x \rightarrow 0^+$

1.7) For $f(x) = x^3 - 5x^2 + 3x + 1$,

$f'(x) = 3x^2 - 10x + 3$

so $f'(-8) = 3 \times 64 + 10 \times 8 + 3 > 0$: $f'(-8) \neq 0$

and $f'(3) = 3 \times 9 - 10 \times 3 + 3 = 0$

$f'(1/3) = 3 \times 1/9 - 10 \times 1/3 + 3 = \frac{1}{3} + 3 - \frac{10}{3} = 0$

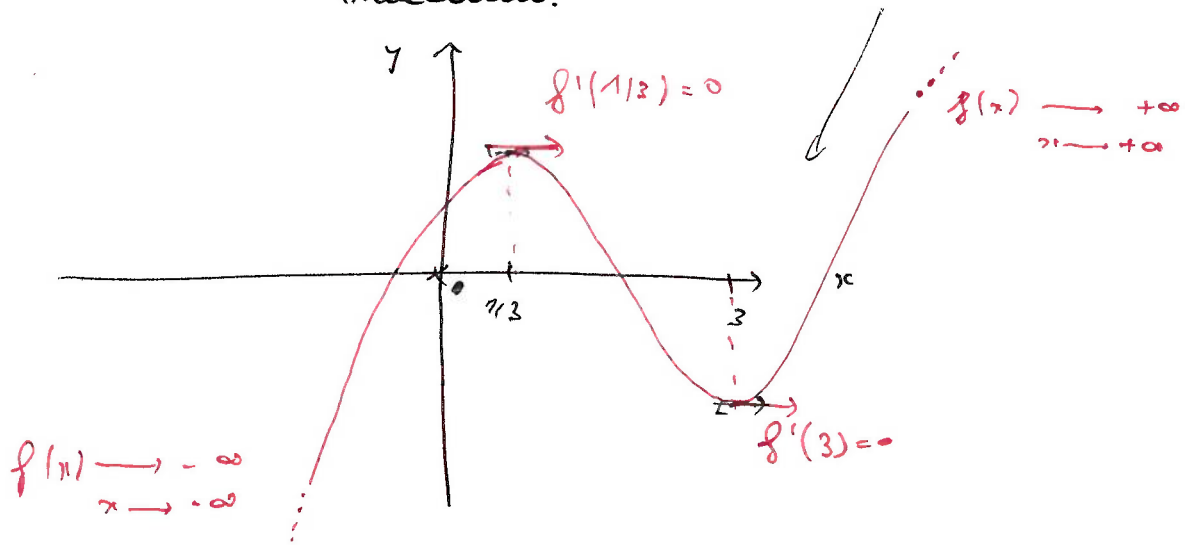
so 3 and 1/3 are the two extreme points of f .

And f is a polynomial of degree 3, so $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ (2)

Problem 1

1.7) ... And $f(x) \rightarrow +\infty$
 $x \rightarrow +\infty$

ie the local minimum is after the local maximum:

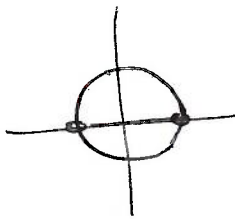


So $1/3$ is the local maximum,
 3 is the local minimum.

Answer C

1.8) Answer C thanks to the lectures.

1.9) f is well defined when $x/2 \neq 0 + \pi k, k \in \mathbb{Z}$
 ie $x \neq 2k\pi, k \in \mathbb{Z}$



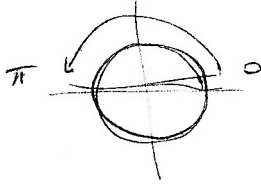
f is defined on $\mathbb{R} \setminus 2\pi\mathbb{Z}$.

On this domain, $f'(x) = \frac{-\frac{1}{2} \cos(x/2)}{(\sin(x/2))^2}$

and $f''(x) = \frac{-\frac{1}{2} \sin(x/2) \cdot (\sin(x/2))^2 - \cos(x/2) \cdot \frac{1}{2} \cdot 2 \cdot \cos(x/2) \sin(x/2)}{(\sin(x/2))^4}$
 $= +\frac{1}{2} \frac{\sin^2(x/2) + \cos^2(x/2)}{(\sin(x/2))^3}$

Problem 1

1.9) ... $f''(x) \geq 0 \iff \sin(x/2) \geq 0$



$\iff x/2 \in (0, \pi) + 2k\pi$
 $\iff x \in (0, 2\pi) + 4k\pi$

So f is concave on $(0, 2\pi) + 4k\pi$

Answer:

(D)

1.10) f is obviously not concave defined: (B)

its domain is too big.

($f(x)$ is not defined if $x \leq \ln(\frac{6}{13}) \dots$)

1.11) We use the change of variable $t = \sqrt{x}$

$x: 1 \rightarrow 4 \quad ; \quad t: 1 \rightarrow 2$

$\frac{dt}{dx} = \frac{1}{2} \frac{1}{\sqrt{x}} \quad \text{so} \quad 2 dt = \frac{dx}{\sqrt{x}}$

$\int_1^4 \frac{\cos(\sqrt{x})}{\sqrt{x}} dx = 2 \int_1^2 \cos(t) dt = 2 \left[\sin(t) \right]_1^2$

$= 2 (\sin(2) - \sin(1)) \quad \checkmark \quad \text{Answer (A)}$

1.12) Answer (A) by following the hint.

1.13) None of the above: the domain of definition

of $P(x) = \frac{1}{x}$ is $\mathbb{R} \setminus \{0\}$,

of $Q(x) = \frac{1}{x \ln(x)}$ is $(0, 1) \cup (1, +\infty)$

but 2 has to be on the interval of resolution.

Answer (D)

Problem-1

1.14) We use the formula seen in class,

$$x(t) = a * (t - \sin(t))$$

$$y(t) = a * (1 - \cos(t))$$

$$\text{So } x'(t) = a(1 - \cos t)$$

$$y'(t) = a \sin(t)$$

$$\begin{aligned} \text{so } (x'(t))^2 + (y'(t))^2 &= a^2 (1 + \cos^2(t) - 2\cos(t) + \sin^2(t)) \\ &= a^2 \cdot 2 \cdot (1 - \cos(t)) \end{aligned}$$

So the formula for the length of the path gives

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{2} a \int_0^{2\pi} \sqrt{1 - \cos(t)} dt \\ &= -2a \left[\sqrt{2 - 2\cos(t)} \cot\left(\frac{t}{2}\right) \right]_0^{2\pi} = \underline{8a} \end{aligned}$$

so Answer (B)

1.15) We just compute:

$$\vec{v}(t) = (x'(t), y'(t))$$

$$= (-aw \sin(awt), aw \cos(awt))$$

so Answer (B)

□