

Chromatic number and Clique number of tournaments

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joint work with

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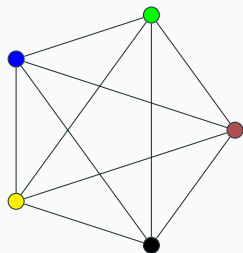
Pierre CHARBIT - IRIF - Université Paris-Cité

Raul Lopes - LIRMM

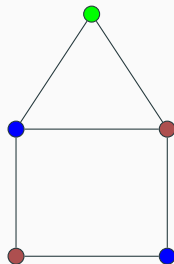
February 16, 2024

The chromatic number

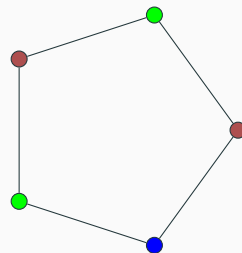
Colouring: adjacent vertices receive distinct colours.



$$\chi = 5$$



$$\chi = 3$$

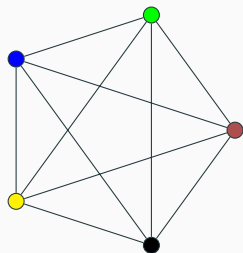


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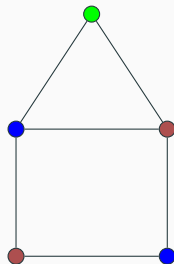
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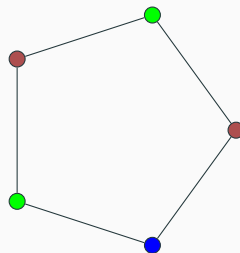
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Chromatic number of $G = \chi(G)$: **minimise** the number of colours.

Complexity : Deciding if $\chi \leq k$ is polynomial for $k = 1, 2$, NP-hard if $k \geq 3$

What does it mean to have **large chromatic number**?

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Equivalently, what families \mathcal{F} are such that $Forb(\mathcal{F})$ has bounded χ ?

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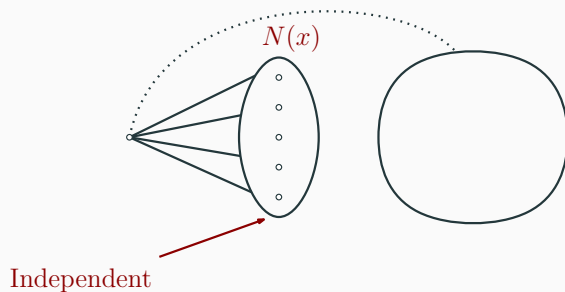
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NO : There exist triangle-free graphs with arbitrarily large χ



$\text{Forb}(\{C_3\})$ has unbounded χ

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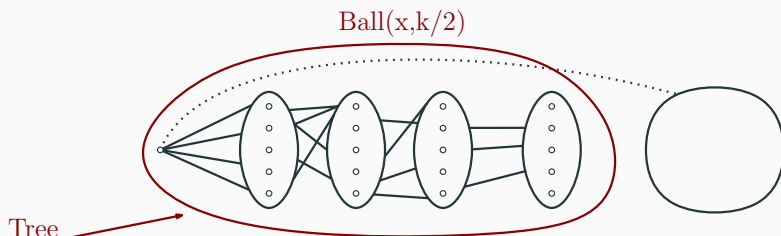
Must it contain a short cycle?

A General question

Why does a graph have **large chromatic number**?

Must it contain a short cycle?

NO : Erdős : There exist graphs with arbitrarily large χ and arbitrarily large girth (shortest cycle length)



$Forb(\{C_3, C_4, \dots, C_k\})$ has unbounded χ

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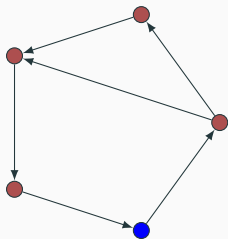
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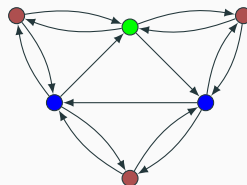
Gyárfás-Sumner Conjecture

Let H be a graph. The class of H -free graphs is χ -bounded if and only if H is a forest.

- Colouring a digraph D : no monochromatic (induced) directed cycle.
- $\vec{\chi}(D)$: the *dichromatic number* of D .



$$\vec{\chi} = 2$$

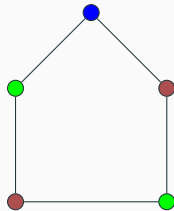


$$\vec{\chi} = 3$$

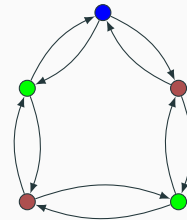
Dichromatic number generalises chromatic number

G : a graph \overleftrightarrow{G} : the digraph obtained by replacing each edge of G by a digon

Theorem: $\chi(G) = \overrightarrow{\chi}(\overleftrightarrow{G})$.



G



\overleftrightarrow{G}

Complexity : Deciding if $\chi \leq k$ is polynomial for $k = 1$, NP-hard if $k \geq 2$

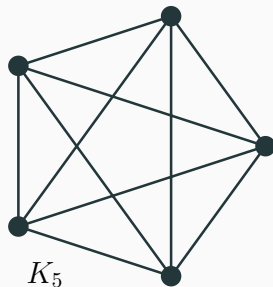
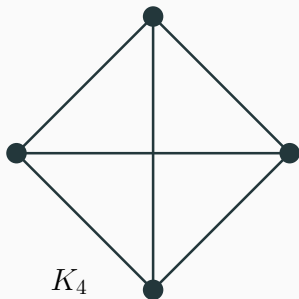
Digraphs with Large $\vec{\chi}$?

For unoriented graphs, graphs with large χ ?

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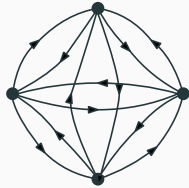
For unoriented graphs, graphs with large χ ?

\Rightarrow Complete graphs $\chi(K_n) = n$



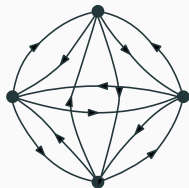
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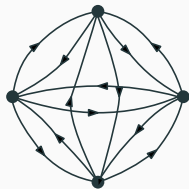
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Any Orientation of complete graphs (aka **tournaments**) ?



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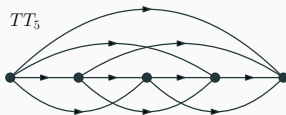


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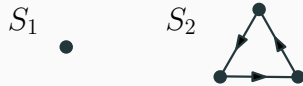


NO because **transitive tournaments** : $\vec{\chi}(TT_k) = 1$



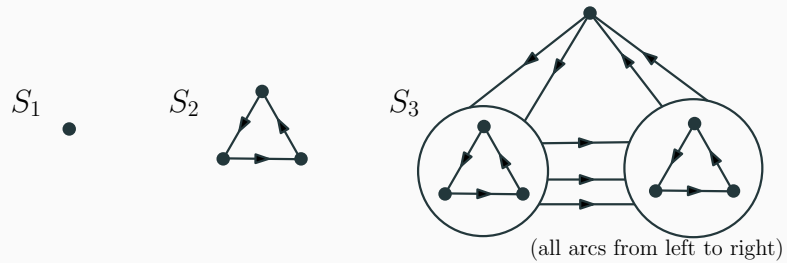
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A recursive construction with $\vec{\chi}(S_k) = k$



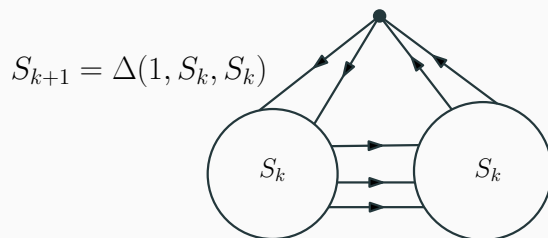
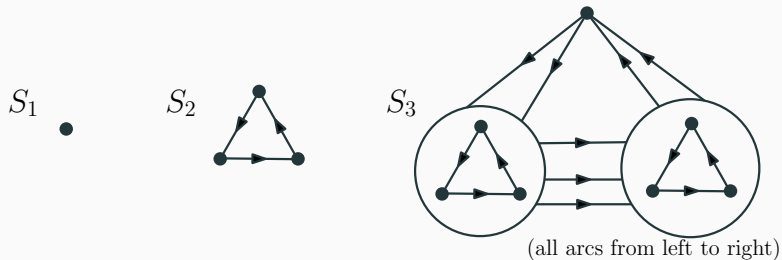
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What is the clique number of a digraph?

We would like that, for every graph G and every digraph D :

$$\omega(G) = \vec{\omega}(\overleftrightarrow{G}) \quad \text{and} \quad \vec{\omega}(D) \leq \vec{\chi}(D)$$

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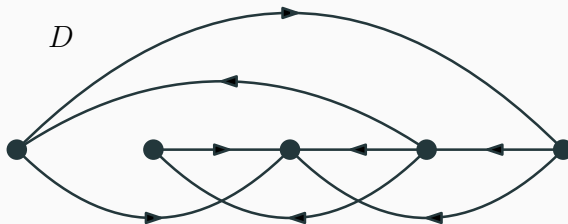
$\vec{\omega}(D)$ = size for a maximum transitive tournament of D .

Interesting, but does not satisfy $\vec{\omega}(D) \leq \vec{\chi}(D)$.

Backedge graph

Let D be a digraph given with a **total order** \prec on $V(D)$.

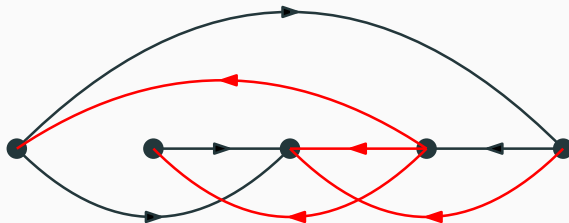
Define the **backedge graph** D^\prec be the **undirected** graph with vertex set $V(D)$ and edge uv if $u \prec v$ and $vu \in A(D)$.



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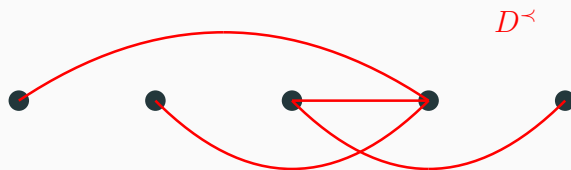
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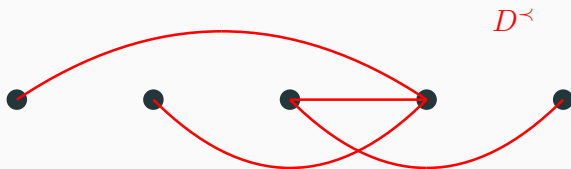
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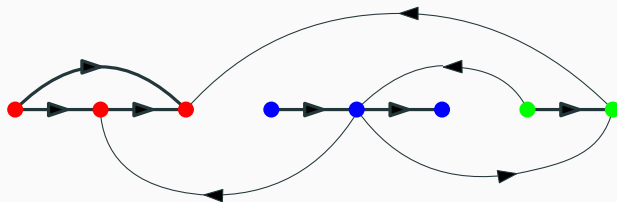
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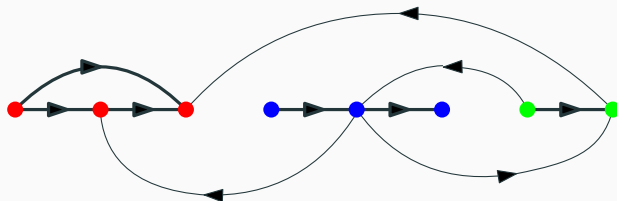
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Moreover, there exists \prec such that $\chi(D^\prec) \leq \vec{\chi}(D)$.

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Moreover, there exists \prec such that $\chi(D^\prec) \leq \vec{\chi}(D)$.

Hence:

$$\vec{\chi}(D) = \min \{ \chi(D^\prec) : \prec \text{ is a total order of } V(D) \}$$

So we have a new definition of the dichromatic number:

$$\vec{\chi}(D) = \min \{ \chi(D^{\prec}) : \prec \text{ is a total order of } V(D) \}$$

This leads a natural definition of the **clique number of a digraph**:

$$\vec{\omega}(D) = \min \{ \omega(D^{\prec}) : \prec \text{ is a total order on } V(D) \}$$

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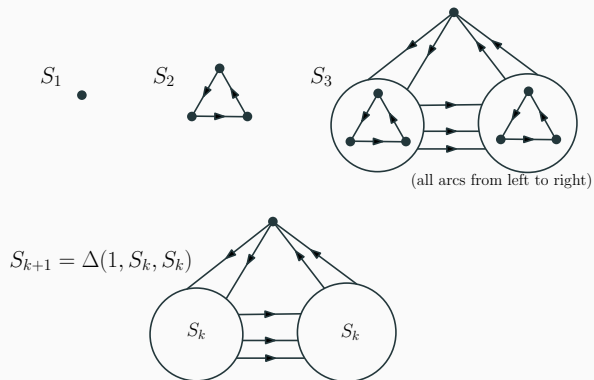
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We clearly have:

- $\vec{\omega}(\overleftrightarrow{G}) = \omega(G)$ (because for every \prec , $\overleftrightarrow{G}^{\prec} = G$), and
- $\vec{\omega}(D) \leq \vec{\chi}(D)$.

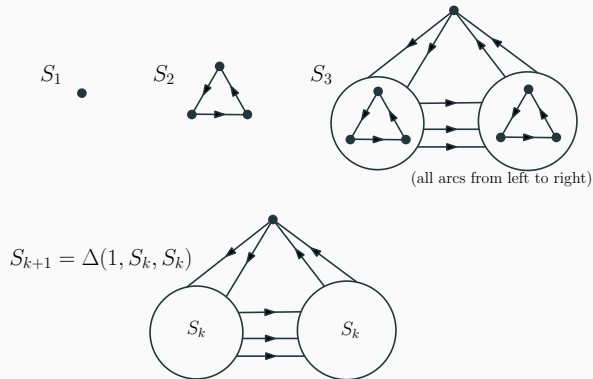
Tournaments with arbitrarily large clique number

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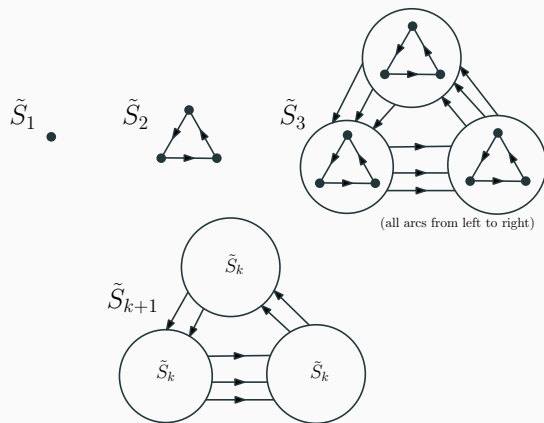


Question

Can we find tournaments with arbitrarily large clique number?

Tournaments with arbitrarily large clique number

Let $\tilde{S}_1 = TT_1$ and inductively, for $n \geq 1$, let $\tilde{S}_n = \Delta(\tilde{S}_{n-1}, \tilde{S}_{n-1}, \tilde{S}_{n-1})$.



Lemma

For any integer n , $\vec{\omega}(\tilde{S}_n) \geq n$.

Theorem (Nguyen, Scott, Seymour, 2023)

For every tournament T and every ordering \prec of $V(T)$.

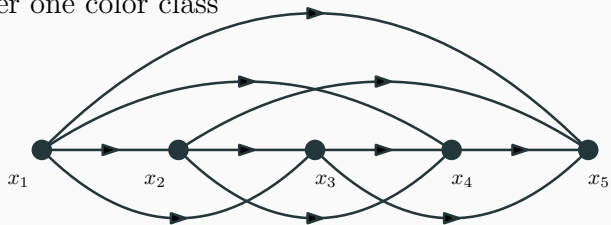
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Consider one color class

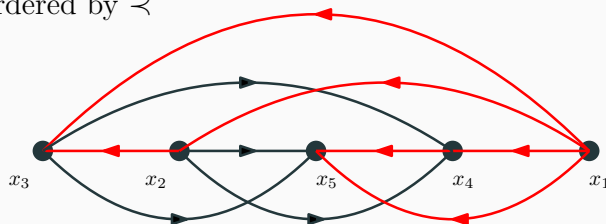


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... ordered by \prec



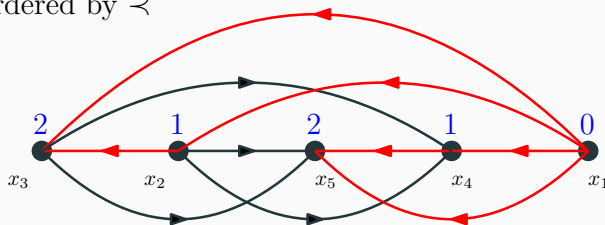
A useful inequality

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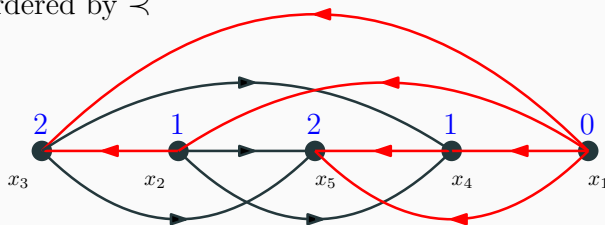
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colour with with "max length back path"

It gives a colouring with at most $\omega(T^{\prec})$ colors

Repeat for the $\vec{\chi}(T)$ color classes

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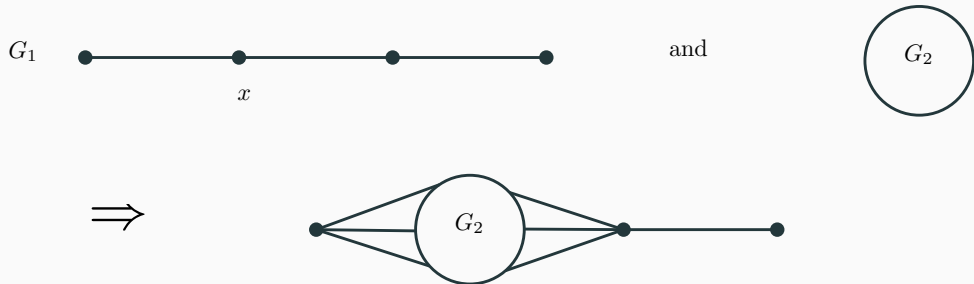
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Example : $\mathcal{C} = \{K_1, K_2\}$

K_1



K_2



$\mathcal{C}^{subst} = \text{cliques}$

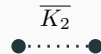


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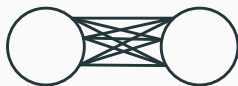
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Example : $\mathcal{C} = \{K_1, K_2, \overline{K_2}\}$



$\mathcal{C}^{subst} = \text{cographs}$
(P_4 -free)



or

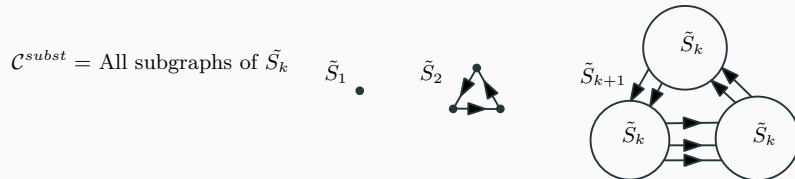
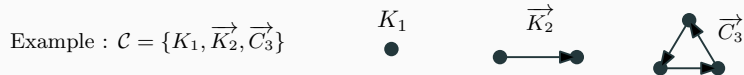


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Conjecture

Let \mathcal{D} be a class of digraphs. If \mathcal{D} is $\vec{\chi}$ -bounded, then so is \mathcal{D}^{subst} .

Theorem

If \mathcal{T} is $\overrightarrow{\chi}$ -bounded, so is \mathcal{T}^{subst}

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Theorem (Chudnovsky, Penev, Scott, Trotignon, 2013)

If a class of *graphs* \mathcal{C} is polynomially χ -bounded, then so is \mathcal{C}^{subst} .

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Theorem (Chudnovsky, Penev, Scott, Trotignon, 2013)

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Question: Is it true that if \mathcal{T} is polynomially $\overrightarrow{\chi}$ -bounded, then so is \mathcal{T}^{subst} ?

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Given a tournament H , $\text{Forb}(H)$ is the class of tournaments T such that T does not contain H as a subtournament.

Question

For which tournament H there exists a function f_H such that :

$$T \in \text{Forb}(H) \Rightarrow \vec{\chi}(T) \leq f(\vec{\omega}(T))$$

We say that such that H are $\vec{\chi}$ -binding.

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Question

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Such tournaments are called **heroes**.

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Question

For which tournament H there exists a number c_H such that

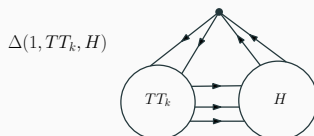
$$T \in \text{Forb}(H) \Rightarrow \vec{\chi}(T) \leq c_H$$

Such tournaments are called **heroes**.

Theorem (Berger Choromanski, Chudnovsky, Fox, Loeb, Scott, Seymour and Thomassé, 2013)

A tournament H is a hero if and only if:

- $H = K_1$.
- $H = (H_1 \Rightarrow H_2)$
- $H = \Delta(1, TT_k, H_1)$ or $H = \Delta(1, H, TT_k)$, where $k \geq 1$ and H_1 is a hero.



Hero

A tournament H is a **hero** if there exists a number c_H such that

$$T \in \text{Forb}(H) \Rightarrow \vec{\chi}(T) \leq c_H$$

Let us define the analogue for $\vec{\omega}$:

Gentlemen

A tournament H is a **gentlemen** if there exists a number c_H such that

$$T \in \text{Forb}(H) \Rightarrow \vec{\omega}(T) \leq c_H$$

Of course, all heroes are gentlemen since $\vec{\omega} \leq \vec{\chi}$

Theorem

Heroes and gentlemen are the same.

Proof:

- We want to prove Gentlemen \Rightarrow Hero (the converse is obvious)
- Take a minimal counter-example H .
- Consider the sequence of tournaments S_1, S_2, S_3, \dots
- We proved that they have arbitrarily large $\vec{\omega}$, so must contain H
- So H is of the form $\Delta(1, A, B)$ (because H must be strong by minimality)
- Nguyen, Scott and Seymour proved $S_3 = \Delta(1, \vec{C}_3, \vec{C}_3)$ is not a gentlemen, so H does not contain S_3
- So one of A or B is a transitive tournament and the other is a gentlemen and thus a hero by induction
- So H is a hero

Recall that we want to characterize those :

A tournament H is $\vec{\chi}$ -bounding if there exists a function f_H such that :

$$T \in \text{Forb}(H) \Rightarrow \vec{\chi}(T) \leq f(\vec{\omega}(T))$$

Theorem

H is $\vec{\chi}$ -bounding $\Rightarrow H$ has an ordering \prec such that H^\prec is a forest.

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- Let H be a tournament such that no backedge graph of H is a forest.
- Let \mathcal{C} be a the class of (undirected) graph with girth at least $|V(H)| + 1$.

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 - Hence, for every $X \subseteq T$ such that $|X| = |V(H)|$, $T^\prec[X]$ is a forest, and thus distinct from H .

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- Observe that every $T \in \mathcal{T}[\mathcal{C}]$ has $\vec{\omega}(T) \leq 2$.

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- Moreover, by a celebrated theorem of Erdős, graph in \mathcal{C} can have arbitrarily large chromatic number.
- Hence, by the Nguyen et al. inequality, there are tournaments in $\mathcal{T}[\mathcal{C}]$ can have arbitrarily large dichromatic number.
- So $\mathcal{T}[\mathcal{C}]$ is not $\vec{\chi}$ -bounded, and thus H -free tournaments is not $\vec{\chi}$ -bounded

Conjecture

Let H be a tournament. $Forb(H)$ is $\overrightarrow{\chi}$ -bounded if and only if H has an ordering \prec for which H^\prec is a forest.

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Recall that:

Gyárfás-Sumner Conjecture, 1981

Let H be a graph. $\text{Forb}(H)$ is χ -bounded if and only if H is a forest.

Gyárfás-Sumner Conjecture for tournaments ?

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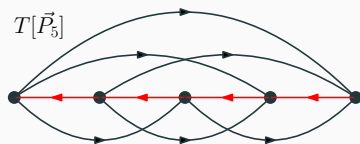
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We proved:

- the only if part (previous slide)
- it is enough to prove it for trees instead of forests,
- If H_1 and H_2 are $\vec{\chi}$ -binding, then so is $H_1 \Rightarrow H_2$,
- It holds for $H = T[\vec{P}_k]$.



Theorem

If H_1 and H_2 are $\vec{\chi}$ -binding, then so it $H_1 \Rightarrow H_2$.

We need the following very nice "local to global" result :

Theorem (Le, Harutyunyan, Thomassé and Wu, 2017)

There exists a function λ such that, if for every vertex v , $\vec{\chi}(T[N^+(v)]) \leq t$, then $\vec{\chi}(T) \leq \lambda(t)$.

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- Say the arc uv is heavy if $\vec{\chi}(T[N(uv)]) \geq 2g(k) + 1$, and light otherwise.



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- For every u ,
$$\min(\vec{\chi}(T[N^-(u)]), \vec{\chi}(T[N^+(u)]) \leq f(k + 1) + g(k) + 2hg(k).$$

□

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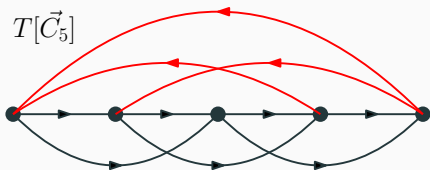
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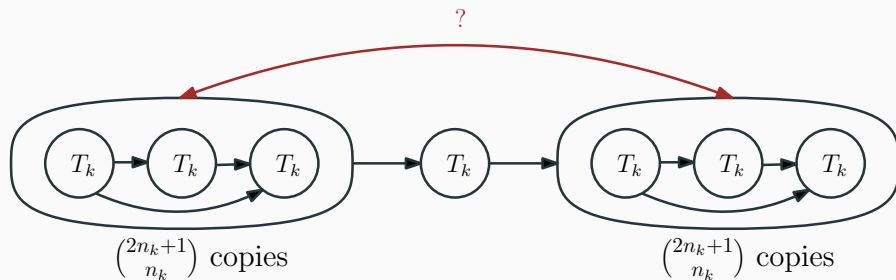
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- $V(D)$ can be partitioned into two sets of bounded dichromatic number

□

A counterexample by G. Aubian



Construction :



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Theorem: given a tournament T , we can decide in poly-time if $\vec{\omega}(T) \geq 3$,
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Arc local to global Theorem [Klingelhoefer and Newman, 2023] If G is an oriented graph such that $\alpha(G) \leq \alpha$, and $N_{\vec{c}_3}(xy) \leq k$, then $\vec{\chi}(G) \leq f(\alpha, k)$.

Some more open questions

Conjecture: The class of tournaments with twinwidth at most k is $\vec{\chi}$ -bounded.

Conjecture (Large \vec{w} implies a \vec{w} -cluster)

There exists two functions f and ℓ such that, for every integer k , every tournament T with $\vec{w}(T) \geq f(k)$ contains a subtournament X with $|X| \leq \ell(k)$ and $\vec{w}(X) \geq k$.

Conjecture: There exists a function g such that, if $\vec{w}(N^+(v)) \leq t$ for every vertex v , then $\vec{w}(T) \leq g(t)$.

Conjecture: for every n -vertex tournament T , $\vec{w}(T) = O(\log(n))$

THANK YOU FOR YOUR ATTENTION