

# Chromatic number and Clique number of tournaments

Guillaume Aubian - Charles University

joint work with

Pierre Aboulker - ENS

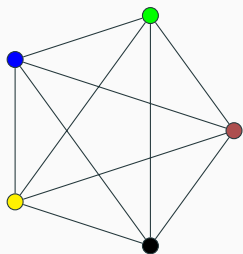
Pierre Charbit - IRIF - Université Paris-Cité

Raul Lopes - LIRMM

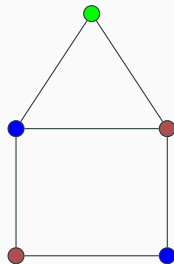
February 16, 2024

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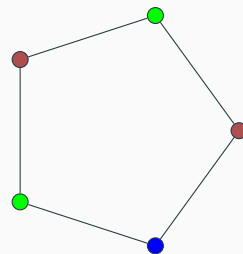
Colouring: adjacent vertices receive distinct colours.



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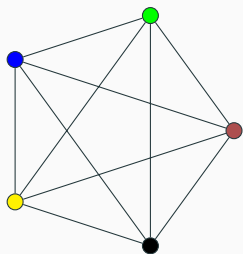


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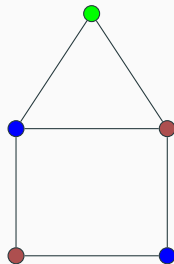
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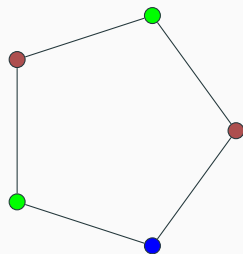
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Deciding if  $\chi(G) \leq k$  is NP-hard if  $k \geq 3$ .

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where  $Forb(\mathcal{F}) = \{G : \forall H \in \mathcal{F}, H \text{ is not a induced subgraph of } G\}$

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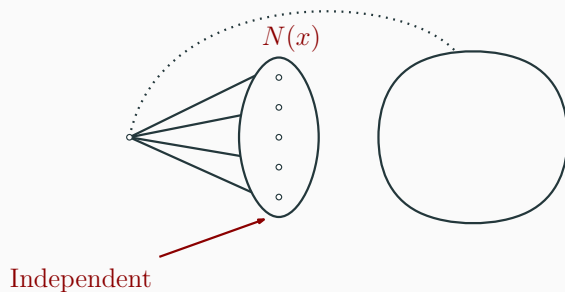
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Must it contain a large clique?

**NO:** There are triangle-free graphs with arbitrarily large  $\chi$



$\text{Forb}(\{C_3\})$  has unbounded  $\chi$

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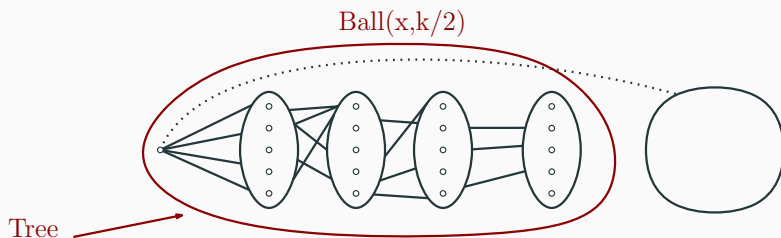
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## A General question

Why does a graph have **large chromatic number**?

Must it contain a short cycle?

**NO:** Erdős proved there exist graphs with arbitrarily large  $\chi$  and arbitrarily large girth (shortest cycle length)



$\text{Forb}(\{C_3, C_4, \dots, C_k\})$  has unbounded  $\chi$

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## Clique number versus chromatic number

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**Answer in some classes:** because it has a large clique  
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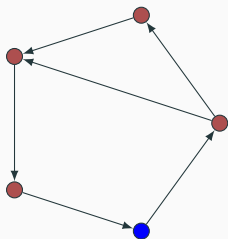
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### Gyárfás-Sumner Conjecture

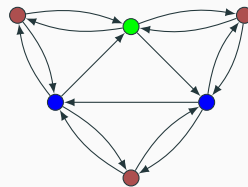
Let  $F$  be a forest.  $Forb_{ind}(F)$  is  $\chi$ -bounded.

- Colouring a digraph  $D$ : no monochromatic (induced) directed cycle.

$$\vec{\chi}(D) = \min\{k \mid D \text{ can be coloured with } k \text{ colours}\}$$



$$\vec{\chi} = 2$$



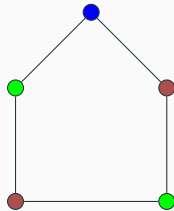
$$\vec{\chi} = 3$$

# Dichromatic number generalises chromatic number

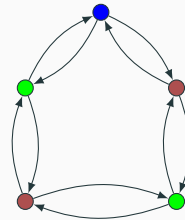
$G$ : a graph

$\overleftrightarrow{G}$ : the digraph obtained by replacing each edge of  $G$  by a digon

**Theorem:**  $\chi(G) = \overrightarrow{\chi}(\overleftrightarrow{G})$ .



$G$

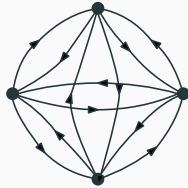


$\overleftrightarrow{G}$

**Complexity** : Deciding if  $\overrightarrow{\chi} \leq k$  is NP-hard whenever  $k \geq 2$ .

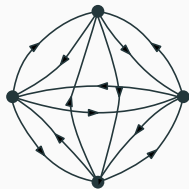
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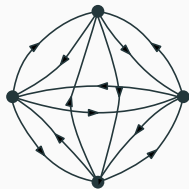
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Does any oriented complete graph (**tournaments**) work?



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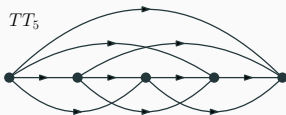


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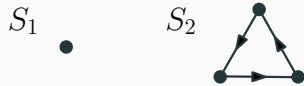


**NO** because **transitive tournaments**:  $\vec{\chi}(TT_k) = 1$



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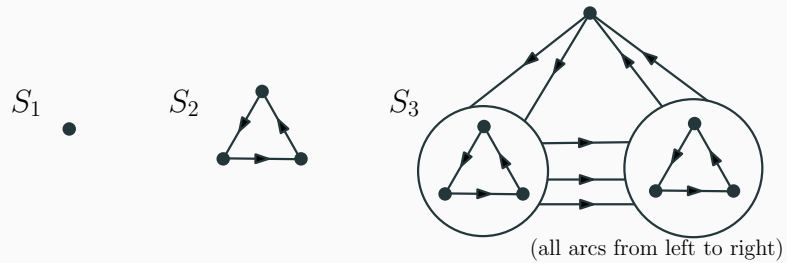
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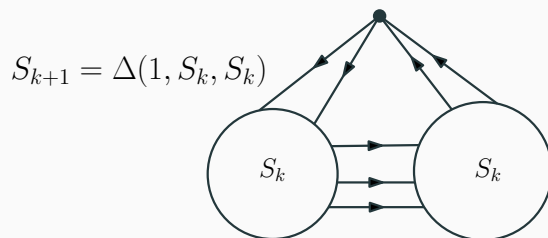
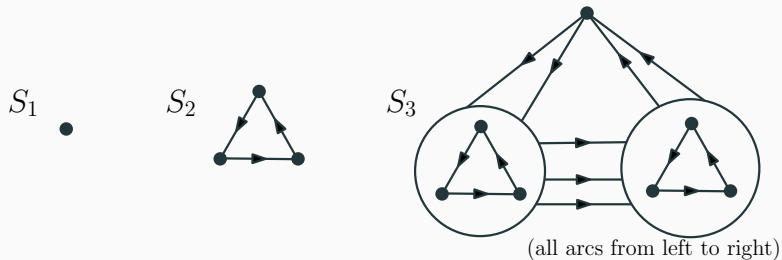
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## What is the clique number of a digraph?

We would like that, for every graph  $G$  and every digraph  $D$ :

$$\omega(G) = \vec{\omega}(\overleftrightarrow{G}) \quad \text{and} \quad \vec{\omega}(D) \leq \vec{\chi}(D)$$

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**Second attempt:**

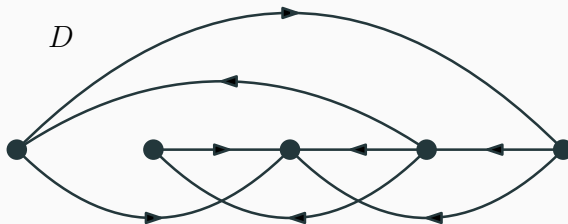
$\vec{\omega}(D)$  = size for a maximum transitive tournament of  $D$ .

Interesting, but does not satisfy  $\vec{\omega}(D) \leq \vec{\chi}(D)$ .

## Backedge graph

Let  $D$  be a digraph and  $\prec$  a **total order** on  $V(D)$ .

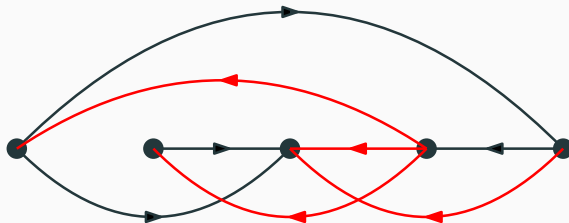
Let the **backedge graph**  $D^\prec$  be the **undirected** graph with vertex set  $V(D)$  and edge  $uv$  if  $u \prec v$  and  $vu \in A(D)$ .



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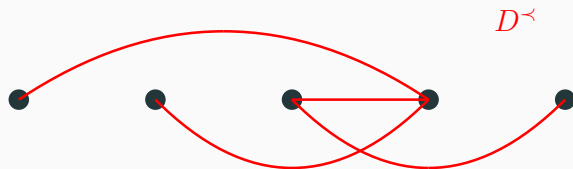
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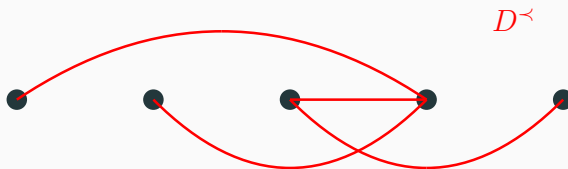




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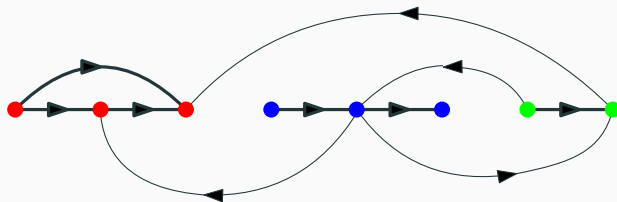
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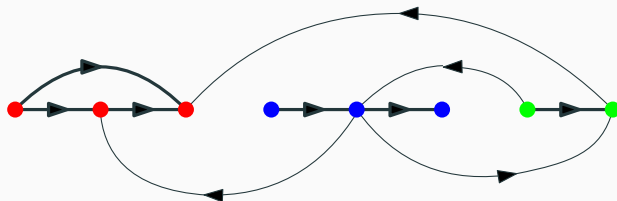
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Moreover, there exists  $\prec$  such that  $\chi(D^\prec) \leq \vec{\chi}(D)$ .

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Moreover, there exists  $\prec$  such that  $\chi(D^\prec) \leq \vec{\chi}(D)$ .

Hence:

$$\vec{\chi}(D) = \min \{ \chi(D^\prec) : \prec \text{ is a total order of } V(D) \}$$

So we have a new definition of the dichromatic number:

$$\vec{\chi}(D) = \min \{ \chi(D^{\prec}) : \prec \text{ is a total order of } V(D) \}$$

This leads to a natural definition of the **clique number of a digraph**:

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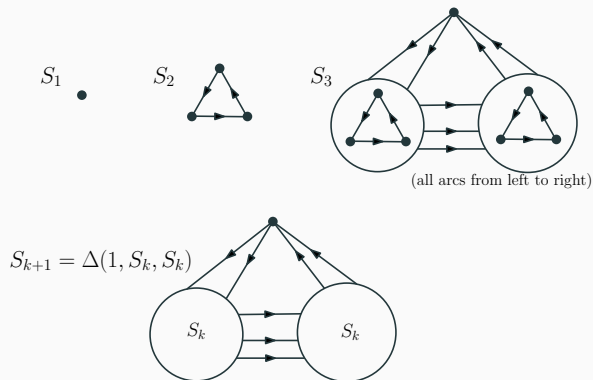
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We clearly have:

- $\vec{\omega}(\overleftrightarrow{G}) = \omega(G)$  (because for every  $\prec$ ,  $\overleftrightarrow{G}^{\prec} = G$ ), and
- $\vec{\omega}(D) \leq \vec{\chi}(D)$ .

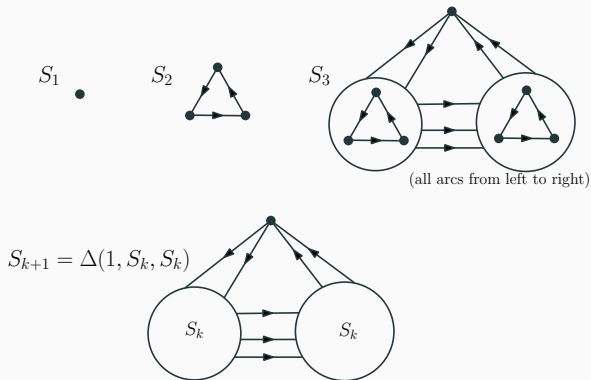
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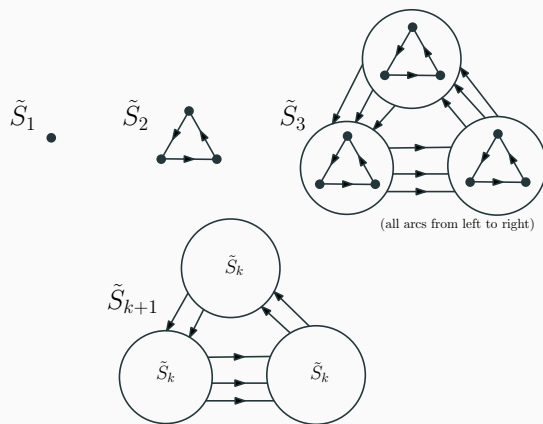


## Question

Can we find tournaments with arbitrarily large clique number?

# Tournaments with arbitrarily large clique number

Let  $\tilde{S}_1 = TT_1$  and inductively, for  $n \geq 1$ , let  $\tilde{S}_n = \Delta(\tilde{S}_{n-1}, \tilde{S}_{n-1}, \tilde{S}_{n-1})$ .



## Lemma

For any integer  $n$ ,  $\vec{\omega}(\tilde{S}_n) \geq n$ .



### Theorem (Nguyen, Scott, Seymour, 2023)

For every tournament  $T$  and every ordering  $\prec$  of  $V(T)$ .

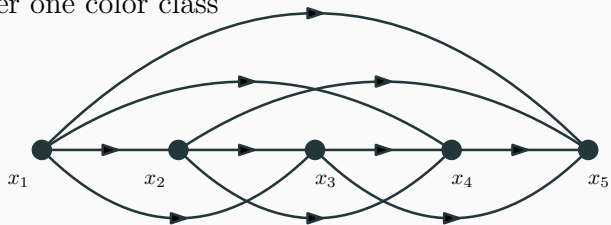
$$\frac{\chi(T^{\prec})}{\omega(T^{\prec})} \leq \vec{\chi}(T) \leq \chi(T^{\prec})$$

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Consider one color class

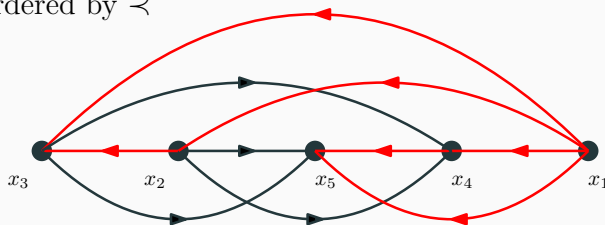


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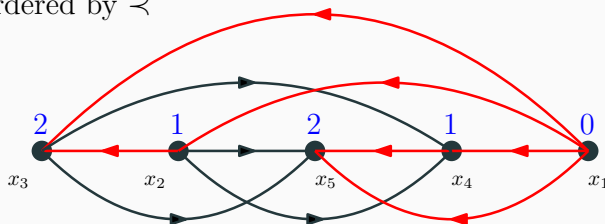
## A useful inequality

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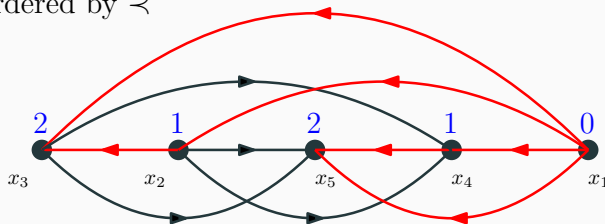
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colour with with "max length back path"

It gives a colouring with at most  $\omega(T^{\prec})$  colors

Repeat for the  $\vec{\chi}(T)$  color classes

A class of tournaments  $\mathcal{T}$  is  $\vec{\chi}$ -bounded if there exists a function  $f$  such that, for every  $T \in \mathcal{T}$ ,

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## Theorem (Folklore)

*If  $\mathcal{C}$  is  $\chi$ -bounded, then so is  $\mathcal{C}^{subst}$*

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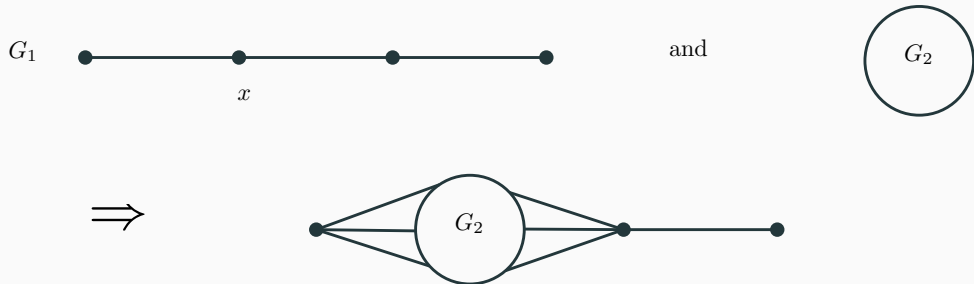
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Example :  $\mathcal{C} = \{K_1, K_2\}$

$K_1$



$K_2$



$\mathcal{C}^{subst} = \text{cliques}$

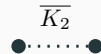


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Example :  $\mathcal{C} = \{K_1, K_2, \overline{K_2}\}$



$\mathcal{C}^{subst} = \text{cographs}$   
( $P_4$ -free)



or

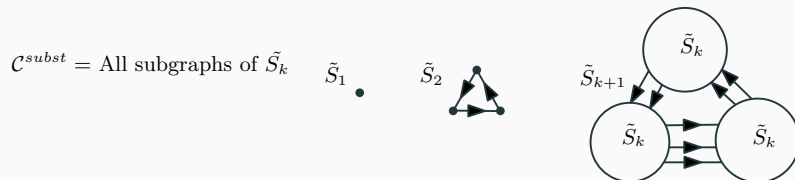


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## Conjecture

Let  $\mathcal{D}$  be a class of digraphs. If  $\mathcal{D}$  is  $\vec{\chi}$ -bounded, then so is  $\mathcal{D}^{subst}$ .



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## Theorem (Chudnovsky, Penev, Scott, Trotignon, 2013)

If a class of *graphs*  $\mathcal{C}$  is polynomially  $\chi$ -bounded, then so is  $\mathcal{C}^{subst}$ .

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Let  $\mathcal{D}$  be a class of digraphs. If  $\mathcal{D}$  is  $\overrightarrow{\chi}$ -bounded, then so is  $\mathcal{D}^{subst}$ .

## Theorem (Chudnovsky, Penev, Scott, Trotignon, 2013)

If a class of *graphs*  $\mathcal{C}$  is polynomially  $\chi$ -bounded, then so is  $\mathcal{C}^{subst}$ .

**Question:** Is it true that if  $\mathcal{T}$  is polynomially  $\overrightarrow{\chi}$ -bounded, then so is  $\mathcal{T}^{subst}$ ?

Given a tournament  $H$ ,  $\text{Forb}(H)$  is the class of tournaments  $T$  such that  $T$  does not contain  $H$  as a subtournament.

## Question

For which tournament  $H$  there exists a function  $f_H$  such that :

$$T \in \text{Forb}(H) \Rightarrow \vec{\chi}(T) \leq f(\vec{\omega}(T))$$

We say that such that  $H$  are  $\vec{\chi}$ -binding.

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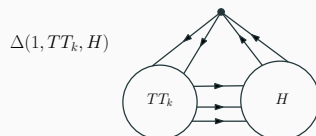
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**Theorem (Berger Choromanski, Chudnovsky, Fox, Loeb, Scott, Seymour and Thomassé, 2013)**

*A tournament  $H$  is a hero if and only if:*

- $H = K_1$ .
- $H = (H_1 \Rightarrow H_2)$
- $H = \Delta(1, TT_k, H_1)$  or  $H = \Delta(1, H, TT_k)$ , where  $k \geq 1$  and  $H_1$  is a hero.



### Hero

A tournament  $H$  is a **hero** if there exists a number  $c_H$  such that

$$T \in \text{Forb}(H) \Rightarrow \vec{\chi}(T) \leq c_H$$

Let us define the analogue for  $\vec{\omega}$  :

### Gentleman

A tournament  $H$  is a **gentleman** if there exists a number  $c_H$  such that

$$T \in \text{Forb}(H) \Rightarrow \vec{\omega}(T) \leq c_H$$

Of course, all heroes are gentlemen since  $\vec{\omega} \leq \vec{\chi}$



## Theorem

*Heroes and gentlemen are the same.*

## Proof:

- We want to prove Gentlemen  $\Rightarrow$  Hero (the converse is obvious)
- Take a minimal counter-example  $H$ .
- Consider the sequence of tournaments  $S_1, S_2, S_3, \dots$
- We proved that they have arbitrarily large  $\vec{\omega}$ , so must contain  $H$
- So  $H$  is of the form  $\Delta(1, A, B)$  (because  $H$  must be strong by minimality)
- Nguyen, Scott and Seymour proved  $S_3 = \Delta(1, \vec{C}_3, \vec{C}_3)$  is not a gentlemen, so  $H$  does not contain  $S_3$
- So one of  $A$  or  $B$  is a transitive tournament and the other is a gentlemen and thus a hero by induction
- So  $H$  is a hero

Recall that we want to characterize those :

A tournament  $H$  is  $\vec{\chi}$ -bounding if there exists a function  $f_H$  such that :

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### Theorem

*$H$  is  $\vec{\chi}$ -bounding  $\Rightarrow H$  has an ordering  $\prec$  such that  $H^\prec$  is a forest.*

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- But it can be proved that  $\chi(G^\prec)$  big and  $\omega(G^\prec)$  small implies  $\vec{\chi}$  small.
- So  $\mathcal{T}[\mathcal{C}]$  is not  $\vec{\chi}$ -bounded.

### Conjecture

Let  $H$  be a tournament.  $Forb(H)$  is  $\overrightarrow{\chi}$ -bounded if and only if  $H$  has an ordering  $\prec$  for which  $H^\prec$  is a forest.

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We need the following very nice "local to global" result :

### Theorem (Le, Harutyunyan, Thomassé and Wu, 2017)

There exists a function  $\lambda$  such that, if for every vertex  $v$ ,  $\vec{\chi}(T[N^+(v)]) \leq t$ , then  $\vec{\chi}(T) \leq \lambda(t)$ .

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- $V(D)$  can be partitioned into two sets of bounded dichromatic number

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**Arc local to global Theorem** [Klingelhoefer and Newman, 2023] If  $G$  is an oriented graph such that  $\alpha(G) \leq \alpha$ , and  $N_{\vec{c}_3}(xy) \leq k$ , then  $\vec{\chi}(G) \leq f(\alpha, k)$ .

## Some more open questions

**Conjecture:** The class of tournaments with twinwidth at most  $k$  is  $\vec{\chi}$ -bounded.

**Conjecture** (Large  $\vec{w}$  implies a  $\vec{w}$ -cluster)

There exists two functions  $f$  and  $\ell$  such that, for every integer  $k$ , every tournament  $T$  with  $\vec{w}(T) \geq f(k)$  contains a subtournament  $X$  with  $|X| \leq \ell(k)$  and  $\vec{w}(X) \geq k$ .

**Conjecture:** There exists a function  $g$  such that, if  $\vec{w}(N^+(v)) \leq t$  for every vertex  $v$ , then  $\vec{w}(T) \leq g(t)$ .

Conjecture: for every  $n$ -vertex tournament  $T$ ,  $\vec{w}(T) = O(\log(n))$

THANK YOU FOR YOUR ATTENTION